Math Prerequisites Homework, Part II

1. Sketch a vector field that has zero divergence everywhere but nonzero curl everywhere, and give a formula for it. Note that a field which circles a point is not the only way to have curl.

2. A student measures $\oint_S \vec{v} \cdot d\vec{A} = 0$ for one particular closed surface S and concludes that $\vec{\nabla} \cdot \vec{v} = 0$ everywhere inside S. Are they right? Prove it or give a counterexample.

3. Consider the field $\vec{v} = \frac{k}{r}\hat{\phi}$, written in cylindrical coordinates. Note that in cylindrical coordinates, the curl of a field with only a ϕ component that depends only on r is given by $\vec{\nabla} \times \vec{f} = \frac{1}{r}\frac{\partial}{\partial r}(r f_\phi)\,\hat{z}$.

   (a) Show that $\vec{\nabla} \times \vec{v} = 0$ everywhere except on the z axis.

   (b) Compute $\oint_C \vec{v} \cdot d\vec{l}$ around a circle of radius a centered on the z axis, traversed counterclockwise.

   (c) A classmate argues that since the curl vanishes everywhere, Stokes' theorem forces the circulation in (b) to be zero, contradicting your answer. Explain how (a) and (b) can both be true, and say precisely where your classmate's argument breaks down. What would you get for a closed loop that does not encircle the z axis?

4. Find the flux of $\vec{v} = k\vec{r}$ out of a cube of side L centered at the origin, with faces perpendicular to the coordinate axes. Do it by direct integration over the faces (do not use the divergence theorem), then check your answer with the divergence theorem.

5. Find the mass of a solid hemisphere of radius R (the half with $z > 0$) whose density is $\rho = k\sqrt{x^2 + y^2}$.

6. A cloud of charge around the origin has density $\rho = \rho_0 e^{-r/a}$ in spherical coordinates.

   (a) Find the total charge. You may integrate out to infinity.

   (b) What fraction of the charge lies inside $r = a$? Comment on whether the answer surprises you.

7. (optional) The integral $\int_{-\infty}^{\infty} e^{-x^2} dx$ cannot be done by any of the one-dimensional techniques you know, because e^{-x^2} has no elementary antiderivative. It can be done in two dimensions.

   (a) Call the integral I. Write I^2 as a single integral over the entire xy plane, then evaluate it in polar coordinates to find I.

   (b) Now evaluate $\int e^{-r^2} d^3r$ over all of space in two different ways: as a product of one-dimensional integrals, and in spherical coordinates. Use the two results together to find $\int_0^{\infty} r^2 e^{-r^2} dr$, which is not otherwise obvious.

   (c) What would the same argument give in d dimensions?