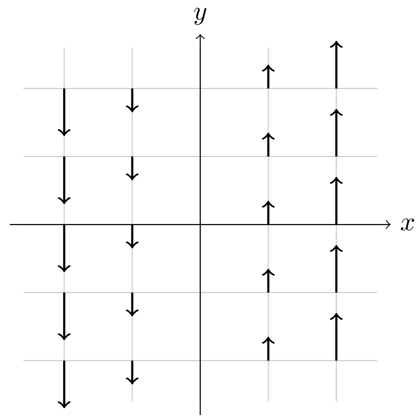
Math Prerequisites Homework, Part II (Solutions)

1. *Sketch a vector field that has zero divergence everywhere but nonzero curl everywhere, and give a formula for it. Note that a field which circles a point is not the only way to have curl.*

   Many solutions are possible. The simplest is a shear, $\vec{v} = x\hat{y}$:

$$\vec{\nabla} \cdot \vec{v} = \frac{\partial(0)}{\partial x} + \frac{\partial(x)}{\partial y} = 0 \qquad \vec{\nabla} \times \vec{v} = \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)\hat{z} = \hat{z}$$



Nothing is piling up anywhere (a small box has as much entering the bottom as leaving the top), but a paddle wheel dropped anywhere would spin, because the arrows on its right are stronger than the arrows on its left.

2. *A student measures $\oint_S \vec{v} \cdot d\vec{A} = 0$ for one particular closed surface S and concludes that $\vec{\nabla} \cdot \vec{v} = 0$ everywhere inside S. Are they right? Prove it or give a counterexample.*

   They are wrong. Zero total flux only says the divergence *integrates* to zero inside; sources and sinks are allowed as long as they cancel.
   Counterexample: take $\vec{v} = z^2\hat{z}$ and let S be a sphere of radius R centered at the origin. The divergence is

$$\vec{\nabla} \cdot \vec{v} = 2z$$

which is nonzero everywhere except the plane $z = 0$. But on the sphere $z = R\cos\theta$ and $\hat{n} \cdot \hat{z} = \cos\theta$, so

$$\oint_S \vec{v} \cdot d\vec{A} = \int_0^{2\pi} \int_0^{\pi} R^2 \cos^2\theta \cos\theta \; R^2 \sin\theta \; d\theta \; d\phi = 2\pi R^4 \int_0^{\pi} \cos^3\theta \sin\theta \; d\theta = 0$$

The top half is a net source and the bottom half is an equal net sink.

On a common alternate answer: many students will propose two equal and opposite point charges placed near each other, both enclosed by S. The physical instinct is exactly right, and it is the picture worth carrying forward, but as stated it does not yet settle the question: the divergence of $\frac{k}{r^2}\hat{r}$ is zero at every point where the field is actually defined, so a student pressed on this can only say "the divergence is zero except at two singular points" – which is the same sentence someone defending the false claim would use. Making the dipole a genuine counterexample requires treating the divergence as a delta function at each charge, which we have not developed.
Give full credit for the dipole if the student notices and states this caveat; otherwise give most of the credit and point out that $\vec{v} = z^2\hat{z}$ is the same idea with the singularities smeared out, so that the source and sink densities are ordinary functions rather than points.

3. Consider the field $\vec{v} = \frac{k}{r}\hat{\phi}$, written in cylindrical coordinates. Note that in cylindrical coordinates, the curl of a field with only a ϕ component that depends only on r is given by $\vec{\nabla} \times \vec{f} = \frac{1}{r}\frac{\partial}{\partial r}(rf_\phi)\,\hat{z}$.

   (a) *Show that $\vec{\nabla} \times \vec{v} = 0$ everywhere except on the z axis.*

   For a field with only a ϕ component depending only on r, the curl is

$$\vec{\nabla} \times \vec{v} = \frac{1}{r}\frac{\partial}{\partial r}(rv_\phi)\,\hat{z} = \frac{1}{r}\frac{\partial(k)}{\partial r}\hat{z} = 0 \qquad (r \neq 0)$$

   (b) *Compute $\oint_C \vec{v} \cdot d\vec{l}$ around a circle of radius a centered on the z axis, traversed counterclockwise.*

   Here $d\vec{l} = a\ d\phi\ \hat{\phi}$ and the field is parallel to it everywhere, so

$$\oint_C \vec{v} \cdot d\vec{l} = \int_0^{2\pi} \frac{k}{a} a\ d\phi = \boxed{2\pi k}$$

   Note this is independent of a, for the same reason the flux was independent of R on the quiz.

   (c) *A classmate argues that since the curl vanishes everywhere, Stokes' theorem forces the circulation in (b) to be zero, contradicting your answer. Explain how (a) and (b) can both be true, and say precisely where your classmate's argument breaks down. What would you get for a closed loop that does not encircle the z axis?*

   The claim in (a) was careful to exclude the z axis, and everything hinges on that. Stokes' theorem needs a surface that is bounded by the loop *and* on which the field is well behaved, but every surface bounded by a circle around the z axis is punctured by that axis, where the field blows up. So the hypothesis fails and the theorem cannot be applied as stated. Curl-free is not the same as conservative unless the region has no holes in it.

   For a loop that stays away from the axis, such a surface does exist, the curl vanishes on it, and the integral is $\boxed{0}$.

   The other way to say this, which we will come back to. Instead of cutting the axis out of the domain, we can keep it and allow the curl to be infinite there in a controlled way. Since all of the circulation comes from an arbitrarily thin neighborhood of the axis, and since we already know from (b) that any loop around it gives $2\pi k$, the curl must be a spike on the axis whose integral over any cross section is $2\pi k$:

$$\vec{\nabla} \times \vec{v} = 2\pi k\ \delta(x)\delta(y)\ \hat{z}$$

   With this reading Stokes' theorem applies with no exceptions and returns $2\pi k$, as it must. This is the exact two-dimensional analogue of $\vec{\nabla} \cdot \left(\frac{\hat{r}}{r^2}\right) = 4\pi\delta^3(\vec{r})$, and with $k = \frac{\mu_0 I}{2\pi}$ it is Ampere's law for a straight wire.

   Note the logic carefully: the coefficient $2\pi k$ was not computed from the field and then substituted into the theorem. It was fixed *by* the requirement that the theorem reproduce the line integral we already did. That is the whole point. The integral theorems are what let us say anything at all about the curl and divergence at a singular source, since no direct differentiation is available there.

4. *Find the flux of $\vec{v} = k\vec{r}$ out of a cube of side L centered at the origin, with faces perpendicular to the coordinate axes. Do it by direct integration over the faces (do not use the divergence theorem), then check your answer with the divergence theorem.*

   The trick is that on any one face the dot product is constant. On the top face, $\hat{n} = \hat{z}$ and

   $$\vec{v} \cdot \hat{n} = k\vec{r} \cdot \hat{z} = kz = \frac{kL}{2}$$

   everywhere on that face, no matter how far out toward the corners you go. The face has area L^2, so it contributes $\frac{kL^3}{2}$. The bottom face has $z = -\frac{L}{2}$ and $\hat{n} = -\hat{z}$, giving the same positive contribution, and by symmetry all six faces match:

   $$\boxed{\Phi = 6\left(\frac{kL^3}{2}\right) = 3kL^3}$$

   Check: $\vec{\nabla} \cdot (k\vec{r}) = 3k$, a constant, so the volume integral is $3kL^3$, which agrees.

5. *Find the mass of a solid hemisphere of radius R (the half with $z > 0$) whose density is $\rho = k\sqrt{x^2 + y^2}$.*

   The density is k times the distance from the z axis, which in *spherical* coordinates is $r\sin\theta$. Since the region is a piece of a sphere, spherical coordinates are worth the slightly uglier integrand:

   $$M = \int_0^{2\pi} \int_0^{\pi/2} \int_0^R (kr\sin\theta)\, r^2 \sin\theta \; dr\; d\theta\; d\phi$$

   The integrals separate:

   $$M = 2\pi k\left(\frac{R^4}{4}\right) \int_0^{\pi/2} \sin^2\theta \; d\theta = 2\pi k \frac{R^4}{4} \cdot \frac{\pi}{4}$$

   $$\boxed{M = \frac{\pi^2 k R^4}{8}}$$

   Sanity check: this is smaller than $\frac{2}{3}\pi R^3 \cdot kR$, as it must be, since kR is the largest the density ever gets.

6. A cloud of charge around the origin has density $\rho = \rho_0 e^{-r/a}$ in spherical coordinates.

   (a) *Find the total charge. You may integrate out to infinity.*

   Nothing depends on angle, so the angular integrals just give 4π and we may use shells of volume $4\pi r^2 dr$:

   $$Q = 4\pi\rho_0 \int_0^\infty r^2 e^{-r/a} \; dr$$

   Two integrations by parts (or a table) give $\int_0^\infty r^2 e^{-r/a} dr = 2a^3$, so

   $$\boxed{Q = 8\pi\rho_0 a^3}$$

   (b) *What fraction of the charge lies inside $r = a$? Comment on whether the answer surprises you.*

   Same integral, stopped at a. Substituting $u = \frac{r}{a}$,

   $$\int_0^a r^2 e^{-r/a} dr = a^3 \int_0^1 u^2 e^{-u} du = a^3 \left[-\left(u^2 + 2u + 2\right)e^{-u}\right]_0^1 = a^3\left(2 - \frac{5}{e}\right)$$

   Dividing by $2a^3$,

   $$\boxed{f = 1 - \frac{5}{2e} \approx 0.080}$$

Only about 8%. Even though the density has already fallen by a factor of e by $r = a$, the r^2 in the volume element means the thin shells further out hold far more charge than the dense core does.

7. *(optional) The integral $\int_{-\infty}^{\infty} e^{-x^2}\,dx$ cannot be done by any of the one-dimensional techniques you know, because e^{-x^2} has no elementary antiderivative. It can be done in two dimensions.*

   (a) *Call the integral I. Write I^2 as a single integral over the entire xy plane, then evaluate it in polar coordinates to find I.*

   Since the name of the integration variable does not matter,

$$I^2 = \left(\int_{-\infty}^{\infty} e^{-x^2}\,dx\right)\left(\int_{-\infty}^{\infty} e^{-y^2}\,dy\right) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)}\,dx\,dy$$

   The integrand depends only on $x^2 + y^2 = r^2$, so this is begging for polar coordinates, and the area element supplies the extra factor of r that makes it doable:

$$I^2 = \int_0^{2\pi}\int_0^{\infty} e^{-r^2}\,r\,dr\,d\phi = 2\pi\left(\frac{1}{2}\right) = \pi$$

$$\boxed{I = \sqrt{\pi}}$$

   The positive root, since the integrand is positive.

   (b) *Now evaluate $\int e^{-r^2}\,d^3r$ over all of space in two different ways: as a product of one-dimensional integrals, and in spherical coordinates. Use the two results together to find $\int_0^{\infty} r^2 e^{-r^2}\,dr$, which is not otherwise obvious.*

   In Cartesian coordinates $e^{-r^2} = e^{-x^2}e^{-y^2}e^{-z^2}$, so the integral factors into three copies of part (a):

$$\int e^{-r^2}\,d^3r = I^3 = \pi^{3/2}$$

   In spherical coordinates nothing depends on angle, so the angular integrals give 4π:

$$\int e^{-r^2}\,d^3r = 4\pi\int_0^{\infty} r^2 e^{-r^2}\,dr$$

   Setting the two equal,

$$\boxed{\int_0^{\infty} r^2 e^{-r^2}\,dr = \frac{\sqrt{\pi}}{4}}$$

   (c) *What would the same argument give in d dimensions?*

   The Cartesian factorization runs identically no matter how many axes there are, so

$$\int e^{-r^2}\,d^d r = \boxed{\pi^{d/2}}$$

   Note that the hard part of this whole problem was choosing coordinates, not integrating. That is going to be the pattern for the rest of the course.