Gravity Homework, Part II **(Solutions)**

Full Name, Period: **due: x/xx**

1. A planet of radius R has uniform density ρ, except that a spherical cave of radius a has been hollowed out of it. The cave is not centered on the planet: its center sits at a position $\vec{d}$ from the planet's center, with $d+a<R$. Do not attempt to integrate anything in this problem.

   (a) *Find the gravitational field at an arbitrary point inside the cave.*

   The trick is that a hollowed sphere is the same thing as a complete sphere plus a smaller sphere of *negative* density $-\rho$ filling the cave. Adding the two mass distributions gives exactly zero density inside the cave and ρ everywhere else, which is the object we want. Since fields superpose, we can add the two fields.

   For a uniform solid sphere the interior field is the one from the tunnel problem,

$$\vec{g} = -\frac{4}{3}\pi G\rho\ \vec{r}$$

   where $\vec{r}$ is measured from that sphere's own center. Measuring from the planet's center, a point in the cave is at $\vec{r}$, and the same point measured from the cave's center is at $\vec{r}-\vec{d}$. So

$$\vec{g} = -\frac{4}{3}\pi G\rho\ \vec{r} + \frac{4}{3}\pi G\rho\left(\vec{r}-\vec{d}\right)$$

$$\boxed{\vec{g} = -\frac{4}{3}\pi G\rho\ \vec{d}}$$

   (b) *Describe your answer in words. Is anything about it surprising?*

   Every $\vec{r}$ cancelled. The field inside the cave is perfectly **uniform**: same magnitude $\frac{4}{3}\pi G\rho d$ and same direction, pointing from the cave's center toward the planet's center, at every single point in the cave.
   This should be surprising. Nothing about the setup is uniform, and the cave can be enormous and wildly off-center, yet an object released anywhere inside it falls in the same direction at the same rate. A student who says only "uniform" has the answer. Saying which direction it points is what separates a real understanding from a lucky cancellation.

   Sanity check: put the cave at the center, $\vec{d}=0$, and the field vanishes, as it must by the shell theorem.

   Worth mentioning when handing these back: superposing a real distribution with a negative copy of what was removed is a general trick, not a one-off. We will use it again later in the year.

2. A moon of radius r and density ρ_m orbits a planet of radius R_p and density ρ_p, at a distance d between their centers. Get too close and the planet's gravity rips the moon apart. We can estimate when with a scaling argument.

   (a) *Consider the two points on the moon nearest to and farthest from the planet. Show that the difference between the planet's gravitational field at those two points is approximately $\frac{4GM_pr}{d^3}$, where M_p is the planet's mass. You will want to expand for $r \ll d$.*

   The near and far points sit at $d-r$ and $d+r$, so

$$\Delta g = \frac{GM_p}{(d-r)^2} - \frac{GM_p}{(d+r)^2} = \frac{GM_p}{d^2}\left[\left(1-\frac{r}{d}\right)^{-2} - \left(1+\frac{r}{d}\right)^{-2}\right]$$

   Expanding each to first order in $\frac{r}{d}$ using $(1\pm x)^{-2} \approx 1 \mp 2x$,

$$\Delta g \approx \frac{GM_p}{d^2}\left[\left(1+\frac{2r}{d}\right)-\left(1-\frac{2r}{d}\right)\right] = \boxed{\frac{4GM_p r}{d^3}}$$

The important feature is the $\frac{1}{d^3}$. Tidal effects die off much faster than gravity itself, which is why they matter only up close. It is also why the whole argument is about a *difference* in field, not the field: a uniform field pulls the moon along without stretching it at all.

(b) *The moon holds itself together with its own gravity. Estimate the field at the moon's surface due to the moon itself, and find the separation d at which the stretching just overcomes it.*

The moon's own surface field is

$$g_m = \frac{Gm}{r^2}$$

Setting the tidal difference equal to it,

$$\frac{4GM_p r}{d^3} = \frac{Gm}{r^2} \quad\Longrightarrow\quad d^3 = \frac{4M_p r^3}{m}$$

$$\boxed{d = r\left(\frac{4M_p}{m}\right)^{1/3}}$$

Any answer of this form is correct. The numerical factor out front depends on exactly which two points you compare and whether you worry about the moon's shape, so a student who gets 2 or 4 or 3 in there has done the physics right.

(c) *Rewrite your answer in terms of the two densities and the planet's radius. What does it say if the two objects have the same density?*

Substituting $M_p = \frac{4}{3}\pi R_p^3 \rho_p$ and $m = \frac{4}{3}\pi r^3 \rho_m$, the r^3 cancels against the r out front:

$$\boxed{d = R_p\left(\frac{4\rho_p}{\rho_m}\right)^{1/3}}$$

The moon's size dropped out entirely. A boulder and a thousand-kilometer moon of the same material are torn apart at the same distance, which is not at all obvious beforehand.

For equal densities $d \approx 1.6R_p$: the danger zone reaches only a little way above the planet's surface. This is the Roche limit, and it is the reason Saturn has rings rather than another moon. The rings sit inside roughly 2.3 Saturn radii. Our estimate is in the right neighborhood; the real calculation gives a larger coefficient because a fluid moon deforms as it is stretched, which makes things worse.

3. Before nuclear physics existed, the best available guess was that the Sun shines by slowly contracting, converting gravitational potential energy into heat and light. Let us see how well that works. Take the Sun to be a uniform sphere with $M_\odot = 1.99 \times 10^{30}$ kg and $R_\odot = 6.96 \times 10^{8}$ m, radiating at $L_\odot = 3.83 \times 10^{26}$ W.

   (a) *Find the gravitational potential energy of a uniform sphere of mass M and radius R, by imagining it assembled shell by shell from material brought in from infinity.*

   Suppose the sphere has been built up to radius x, so the mass already in place is $m(x) = \frac{4}{3}\pi\rho x^3$. Bringing in the next shell, of thickness dx and mass $dm = 4\pi\rho x^2 dx$, from infinity releases

$$dU = -\frac{Gm(x)\,dm}{x} = -\frac{16\pi^2 G\rho^2}{3}x^4 dx$$

   Integrating out to R,

$$U = -\frac{16\pi^2 G\rho^2}{3} \cdot \frac{R^5}{5} = -\frac{16\pi^2 G\rho^2 R^5}{15}$$

   Substituting $\rho = \frac{3M}{4\pi R^3}$,

$$\boxed{U = -\frac{3GM^2}{5R}}$$

   The shell already in place pulls on the incoming shell as though all of it were at the center, which is what lets us write $\frac{Gm(x)dm}{x}$ without integrating over the interior. That is the shell theorem doing real work.

   (b) *How long could the Sun shine at its present brightness on that energy alone?*

$$|U| = \frac{3\left(6.674 \times 10^{-11}\right)\left(1.99 \times 10^{30}\right)^2}{5\left(6.96 \times 10^{8}\right)} \approx 2.3 \times 10^{41}\ \text{J}$$

$$t = \frac{|U|}{L_\odot} \approx \frac{2.3 \times 10^{41}}{3.83 \times 10^{26}} \approx 6 \times 10^{14}\ \text{s}$$

$$\boxed{t \approx 2 \times 10^{7}\ \text{years}}$$

   Anything within a factor of two counts. Students who forget that only some fraction of the released energy is radiated, or who use $\frac{GM^2}{R}$ without the $\frac{3}{5}$, land in the same place, which is the point of an estimate.

   (c) *The Earth is about 4.5×10^{9} years old and there is fossil evidence of life for most of that time. What does your answer imply, and what is the resolution?*

   The Sun could only last about 20 million years on gravitational energy, which is roughly **200 times too short**. The Sun would have had to be far younger than the rocks and fossils lying around on Earth.

   This was a genuine crisis in the nineteenth century, with physicists (Kelvin, Helmholtz) and geologists flatly contradicting each other, and each side fairly confident the other had blundered.

   The resolution is that gravity is not the Sun's energy source at all: nuclear fusion is, and it releases far more energy per kilogram than gravitational contraction can. Nobody could have guessed this, since the necessary physics did not exist yet.

   Full credit for identifying the contradiction and naming fusion. The historical detail is a bonus.

   Worth saying out loud: the calculation was not wrong. It was a correct answer to the question "how long can gravity alone power the Sun," and its disagreement with the rocks is exactly how we learned that something else was going on.

4. Our own galaxy has a flat rotation curve, like the one on the previous homework. The Sun sits about 8 kpc from the galactic center and orbits at roughly 220 km/s. One parsec is 3.09×10^{16} m, and $M_\odot = 1.99 \times 10^{30}$ kg.

   (a) *Estimate the mass of the galaxy interior to the Sun's orbit, in solar masses.*

   A circular orbit needs

$$\frac{v^2}{r} = \frac{GM(r)}{r^2} \implies M(r) = \frac{v^2 r}{G}$$

   With $r = 8\text{ kpc} = 2.47 \times 10^{20}$ m and $v = 2.2 \times 10^5$ m/s,

$$M = \frac{\left(2.2 \times 10^5\right)^2 \left(2.47 \times 10^{20}\right)}{6.674 \times 10^{-11}} \approx 1.8 \times 10^{41}\text{ kg}$$

$$\boxed{M \approx 9 \times 10^{10}\ M_\odot}$$

   Note how little we needed: one speed, one radius, and the assumption of a circular orbit. We never had to see the mass, or even know what it is made of.

   (b) *The flat rotation curve continues out to at least* 50 *kpc. Estimate the mass interior to that radius. What fraction of it lies inside the Sun's orbit?*

   Since v is constant, $M(r) \propto r$, so the mass scales directly with radius:

$$M(50\text{ kpc}) = \frac{50}{8} M(8\text{ kpc}) \approx 6.25 \times 9 \times 10^{10}$$

$$\boxed{M \approx 5.6 \times 10^{11}\ M_\odot}$$

   Only about 16% of that mass lies inside the Sun's orbit.
   Now compare with what we can see: the galaxy's light is strongly concentrated toward the center, and there is very little of it out at 50 kpc. Almost all of the mass is out where almost none of the light is.

   (c) *Suppose the flat rotation curve continued forever. What would that imply about the total mass of the galaxy? Is that acceptable, and what must actually happen?*

   If v is constant forever then $M(r) = \frac{v^2 r}{G}$ grows without bound, so the galaxy would have **infinite total mass**.
   That is not acceptable, so the flat part must end somewhere. The rotation curve has to eventually turn over and fall off like $\frac{1}{\sqrt{r}}$, which is what happens once you are outside essentially all of the mass and the galaxy starts behaving like a point.
   The awkward part, and the reason this is still an active research area, is that nobody has cleanly observed where that turnover happens for most galaxies. We run out of tracers, since there is little visible material that far out to measure speeds from.

   A correct answer identifies the divergence and says the curve must eventually fall. Getting $\frac{1}{\sqrt{r}}$ is a bonus.

5. A science fiction novel describes a hollow planet: a thick spherical shell of total mass M with an empty region of radius a inside it, where the characters live standing on the inner surface with the sky arching overhead.

   (a) *What is the gravitational field everywhere inside the hollow region? What does this mean for the characters?*

   Take a spherical Gaussian surface anywhere inside the hollow region. It encloses no mass at all, and the symmetry of the situation means g is the same everywhere on it and points radially, so

$$\boxed{\vec{g} = 0}$$

everywhere inside, not just at the center.

The characters would not stand on the inner surface. They would **float**, weightless, along with everything else in there. The moment one of them pushed off a wall they would drift until they hit the opposite side. The novel's premise does not work.

(b) *Would making the shell more massive, or much thicker, change your answer?*

$\boxed{\text{No.}}$ The Gaussian surface still encloses nothing, whatever is going on outside it.

This is worth pausing on, because it is genuinely strange: the shell is pulling on you the whole time, hard, from every direction. The pulls simply cancel exactly. Doubling the mass doubles every one of those pulls, and they still cancel.

Students often expect the nearer part of the shell to win. It does pull harder, going as $\frac{1}{r^2}$, but there is correspondingly less of it in that direction, going as r^2, and the two effects cancel exactly. That exact cancellation is special to an inverse square law.

(c) *Suppose the shell were not quite spherical, say slightly egg-shaped. Would the field inside still vanish? You do not need to calculate anything.*

$\boxed{\text{No, not in general.}}$ The cancellation in part (b) depended on the geometry being exactly spherical, so that the extra distance to the far side compensated precisely for the extra amount of material there. Deform the shell and that balance is broken, leaving a nonzero field inside.

Problem 1 on this assignment is a concrete example: the hollow region there had a perfectly good nonzero field in it, uniform and pointing toward the planet's center, precisely because the surrounding material was not distributed symmetrically about the hollow.

A correct answer says the field no longer vanishes and ties it to the loss of spherical symmetry. Guessing "no" with no reason does not count.