Forces and Kinematics: Extra Practice (Solutions)

These are for practice and are not collected. Every answer should be in symbols.

1. *An object of mass m is released from rest and falls under gravity through air that exerts a drag force $\vec{F}_d = -b\vec{v}$, where $b > 0$.*

   (a) *Without solving anything, find the terminal speed v_T and the only combination of m, b and g that has units of time. Call it τ.*

   At terminal speed the acceleration is zero, so the two forces balance: $mg = bv_T$, giving

   $$\boxed{v_T = \frac{mg}{b}} \qquad \boxed{\tau = \frac{m}{b}}$$

   b has units of $\frac{\text{kg}}{\text{s}}$, so $\frac{m}{b}$ is the only time available; note that g cannot appear in it. Everything below will turn out to be a function of $\frac{t}{\tau}$ alone, which is worth expecting before doing any work.

   (b) *Find $v(t)$, taking downward as positive.*

   $$m\frac{dv}{dt} = mg - bv$$

   Separate and integrate from rest:

   $$\int_0^v \frac{dv'}{g - \frac{b}{m}v'} = \int_0^t dt' \qquad \Longrightarrow \qquad -\frac{m}{b}\ln\left(1 - \frac{bv}{mg}\right) = t$$

   $$\boxed{v(t) = \frac{mg}{b}\left(1 - e^{-bt/m}\right) = v_T\left(1 - e^{-t/\tau}\right)}$$

   (c) *Find $x(t)$, measuring x downward from the release point.*

   Integrate again:

   $$\boxed{x(t) = v_T\left[t - \tau\left(1 - e^{-t/\tau}\right)\right]}$$

   At late times this approaches $v_T\,(t - \tau)$: a straight line, lagging behind the no-drag-onset line by exactly one time constant.

   (d) *Show that for $t \ll \tau$ your answers reduce to the familiar free-fall results, and explain physically why they must.*

   Expanding $e^{-t/\tau} = 1 - \frac{t}{\tau} + \frac{1}{2}\frac{t^2}{\tau^2} - \ldots$,

   $$v \approx v_T\left(\frac{t}{\tau} - \frac{t^2}{2\tau^2}\right) = gt\left(1 - \frac{t}{2\tau}\right) \to gt$$

   $$x \approx v_T\left(\frac{t^2}{2\tau}\right) = \frac{1}{2}gt^2$$

   Physically: drag is proportional to speed, and immediately after release the speed is nearly zero, so for a short time there is effectively no drag at all. Every drag problem starts out looking like free fall, and the first correction, $-\frac{gt^2}{2\tau}$, tells you how long "short" is.

2. *A puck of mass m slides on frictionless ice with initial speed v_0. The only horizontal force is air resistance. Consider two models for it: a linear drag $F = -bv$ and a quadratic drag $F = -cv^2$.*

(a) *For each model, find $v(t)$.*

Linear. $m\frac{dv}{dt} = -bv$ separates to $\frac{dv}{v} = -\frac{b}{m}dt$, giving

$$\boxed{v(t) = v_0 e^{-bt/m}}$$

Quadratic. $m\frac{dv}{dt} = -cv^2$ separates to $\frac{dv}{v^2} = -\frac{c}{m}dt$, and integrating gives $\frac{1}{v} - \frac{1}{v_0} = \frac{c}{m}t$, so

$$\boxed{v(t) = \frac{v_0}{1 + \frac{cv_0}{m}t}}$$

Note the difference in character already: exponential decay versus a power law. Neither ever reaches zero, so in both models the puck never formally stops.

(b) *For each model, find the total distance traveled before the puck stops. Comment on the comparison.*

Linear. $x(t) = \frac{mv_0}{b}\left(1 - e^{-bt/m}\right)$, so as $t \to \infty$

$$\boxed{x_\infty = \frac{mv_0}{b}}$$

a perfectly finite distance, even though it takes forever to get there.

Quadratic. $x(t) = \frac{m}{c}\ln\left(1 + \frac{cv_0}{m}t\right)$, which

$$\boxed{\text{diverges}}$$

The puck travels arbitrarily far.

This is the point of the problem, and it is genuinely surprising the first time. Both models slow the puck forever without stopping it, but only one of them keeps it in a bounded region. The reason is what happens at *small* v: there, $v^2 \ll v$, so quadratic drag is much weaker than linear drag, and the long slow tail of the motion is barely damped at all. The behavior of a force law near zero can matter more than its behavior at large values.

(c) *For the quadratic model, find $v(x)$ directly, without using your answer to (a).*

Trade t for x using $\frac{dv}{dt} = v\frac{dv}{dx}$, which is the standard move whenever the force depends on position or whenever you want position rather than time:

$$mv\frac{dv}{dx} = -cv^2 \qquad \Longrightarrow \qquad \frac{dv}{v} = -\frac{c}{m}dx$$

$$\boxed{v(x) = v_0 e^{-cx/m}}$$

The decay is exponential in *space* rather than in time, and it confirms (b) independently: v is nonzero for every finite x, so no finite stopping distance exists.

3. *A uniform rope has mass M and length L.*

(a) *The rope hangs at rest from a ceiling. Find the tension as a function of the distance y measured upward from the bottom end.*

Cut the rope at height y and draw a box around the piece below the cut. That piece has mass $\frac{M}{L}y$, it is in equilibrium, and the only things touching it are gravity and the tension at the cut:

$$\boxed{T(y) = \frac{Mg}{L}y}$$

Linear from zero at the free bottom end to Mg at the ceiling. The tension in a rope is not one number unless the rope is massless.

(b) *The rope now lies on a frictionless floor and is pulled by a horizontal force F applied to one end. Find the tension a distance x from the pulled end.*

The whole rope accelerates together at $a = \frac{F}{M}$. Cut at x and consider the trailing piece, of mass $\frac{M}{L}(L-x)$; the only horizontal force on it is the tension at the cut:

$$T(x) = \frac{M}{L}(L-x)\frac{F}{M}$$

$$\boxed{T(x) = F\left(1 - \frac{x}{L}\right)}$$

Same linear profile as the hanging rope, for the same reason: at any cut, the tension has to accelerate everything behind it. In (a) that meant supporting a weight; here it means accelerating a mass.

(c) *A block of mass M_b is now tied to the far end of that rope, and F is applied as before. Find the tension at both ends of the rope, and show that the massless-rope result is recovered in the appropriate limit.*

Now $a = \frac{F}{M+M_b}$. At the block end the tension accelerates only the block; at the pulled end it accelerates the block plus the whole rope:

$$\boxed{T_{block} = \frac{M_b F}{M + M_b}} \qquad \boxed{T_{pulled} = F}$$

and in general $T(x) = \left[M_b + \frac{M}{L}(L-x)\right]\frac{F}{M+M_b}$.

As $M \to 0$ both ends give $T \to F$ and the x dependence disappears. This is the justification for the usual assumption that "the tension is the same throughout," and it shows exactly what is being assumed: not that the rope is ideal in some vague sense, but specifically that its mass is negligible compared to what it is pulling.

4. *A block of mass m rests on the frictionless inclined face of a wedge of mass M, which sits on a frictionless floor. The incline makes an angle θ with the horizontal. A horizontal force F is applied to the wedge.*

(a) *Find the F for which the block does not slide relative to the wedge.*

If the block does not slide, then block and wedge move as one object, so $a = \frac{F}{M+m}$. Now isolate the block. Only two forces act on it: gravity, and the normal force N perpendicular to the incline face. Writing components in ground-frame horizontal and vertical axes (not axes along the incline — the acceleration is horizontal, so horizontal axes are the convenient ones here),

$$N\sin\theta = ma \qquad N\cos\theta = mg$$

Dividing kills N and m at once:

$$\tan\theta = \frac{a}{g} \qquad \Longrightarrow \qquad a = g\tan\theta$$

$$\boxed{F = (M+m)\,g\tan\theta}$$

Incidentally $N = \frac{mg}{\cos\theta}$, which is *larger* than the $mg\cos\theta$ of a stationary incline.

(b) *Redo the argument in the reference frame of the wedge, and state the condition in a single sentence about "effective gravity."*

In the wedge frame the block is in equilibrium, but there is a pseudoforce ma pointing backward (opposite the acceleration). Combining it with gravity gives an effective gravity of magnitude $m\sqrt{g^2+a^2}$, tilted from the vertical by $\arctan\frac{a}{g}$ toward the back of the wedge.

The condition in one sentence: the block stays put exactly when the effective gravity points perpendicular to the incline surface, since a frictionless surface can only push perpendicular to itself and so can only balance a force in that direction. That requires the tilt angle to equal θ, which is $a = g \tan\theta$ again.

(c) *Check the limits $\theta \to 0$ and $\theta \to \frac{\pi}{2}$, and say in words what each is telling you.*

$\theta \to 0$: $F \to 0$. The face is horizontal and the block will sit on it happily with no push at all — and indeed any nonzero F would leave it behind, since nothing could accelerate it.

$\theta \to \frac{\pi}{2}$: $F \to \infty$. The face is a vertical wall, and holding a block against a frictionless vertical wall requires an infinite normal force, since only friction could support its weight and there is none. The formula is telling you the task is impossible rather than merely difficult.

5. *A curve of radius R is banked at an angle θ. The coefficient of static friction between the tires and the road is μ.*

(a) *Find the one speed v_0 at which the curve can be taken with no friction at all.*

With no friction the only forces are gravity and the normal force. Vertically $N\cos\theta = mg$; horizontally the centripetal requirement is $N\sin\theta = \frac{mv_0^2}{R}$. Dividing,

$$\boxed{v_0 = \sqrt{gR\tan\theta}}$$

Note that m cancels, so the ideal speed is the same for a motorcycle and a loaded truck.

(b) *Find the maximum speed for which the car does not slide.*

Above the ideal speed the car tends to slide *up* the bank, so friction acts down the incline with magnitude μN at the limit. Resolving into horizontal and vertical components:

$$\text{vertical}: \qquad \mathrm{N}\cos\theta - \mu\mathrm{N}\sin\theta = \mathrm{mg}$$

$$\text{horizontal}: \qquad \mathrm{N}\sin\theta + \mu\mathrm{N}\cos\theta = \frac{\mathrm{mv}^2}{\mathrm{R}}$$

Divide the second by the first — N and m both disappear — and then divide numerator and denominator by $\cos\theta$:

$$\frac{v^2}{gR} = \frac{\sin\theta + \mu\cos\theta}{\cos\theta - \mu\sin\theta} = \frac{\tan\theta + \mu}{1 - \mu\tan\theta}$$

$$\boxed{v_{max} = \sqrt{gR\,\frac{\tan\theta + \mu}{1 - \mu\tan\theta}}}$$

(c) *Find the minimum speed, and the condition on μ and θ under which there is no minimum at all.*

Below the ideal speed the car tends to slide *down* the bank, so friction reverses and the whole calculation goes through with $\mu \to -\mu$:

$$\boxed{v_{min} = \sqrt{gR\,\frac{\tan\theta - \mu}{1 + \mu\tan\theta}}}$$

If $\mu \geq \tan\theta$ the numerator is not positive and v_{min} has no real value. The correct reading is not that the formula failed but that there is no minimum: friction alone can hold the car on the bank, so it could sit parked on the curve without sliding. That is exactly the condition for a block not to slide on a stationary incline, which is what a parked car is.

Similarly, v_{max} diverges when $\mu\tan\theta \to 1$, i.e. $\mu \geq \cot\theta$: a bank that steep with tires that good cannot be exited off the top at any speed.

Check: setting $\mu = 0$ collapses both answers to v_0, as it must — with no friction the window of allowed speeds closes to a single point.

6. *A ball of mass m is swung in a vertical circle of radius L.*

(a) *The ball is on the end of a string. Find the minimum speed it can have at the top of the circle, and explain what physically goes wrong below that speed.*

At the top both gravity and the tension point downward, toward the center, so

$$mg + T = \frac{mv_{top}^2}{L}$$

A string can pull but not push, so $T \geq 0$, and the limiting case is $T = 0$:

$$\boxed{v_{top} = \sqrt{gL}}$$

Below this speed the equation would require $T < 0$, i.e. the string would have to push outward, which it cannot do. What actually happens is that the string goes slack and the ball leaves the circular path and becomes a projectile.

(b) *The string is replaced by a rigid rod of negligible mass. How does the answer change, and why?*

The minimum speed at the top becomes $\boxed{v_{top} = 0}$. A rod can push as well as pull, so it can supply an upward force to make up whatever is left over when $\frac{mv^2}{L} < mg$; at $v = 0$ the rod simply holds the ball statically above the pivot. The constraint that produced the answer in (a) was never about circular motion — it was about the one-sided nature of a string.

(c) *With the string at its minimum speed at the top, find the speed and the tension at the bottom.*

Energy conservation over the drop of $2L$:

$$v_{bot}^2 = v_{top}^2 + 4gL = 5gL \qquad \Longrightarrow \qquad \boxed{v_{bot} = \sqrt{5gL}}$$

At the bottom the center is above the ball, so the tension points up and gravity down:

$$T - mg = \frac{mv_{bot}^2}{L} = 5mg \qquad \Longrightarrow \qquad \boxed{T_{bot} = 6mg}$$

(d) *Show that $T_{bot} - T_{top}$ has the same value for any speed, not just the minimum one.*

Write both tensions in general:

$$T_{top} = \frac{mv_{top}^2}{L} - mg \qquad\qquad T_{bot} = \frac{mv_{bot}^2}{L} + mg$$

$$T_{bot} - T_{top} = \frac{m}{L}\left(v_{bot}^2 - v_{top}^2\right) + 2mg$$

Energy conservation gives $v_{bot}^2 - v_{top}^2 = 4gL$ regardless of how fast the ball is going, so

$$\boxed{T_{bot} - T_{top} = 6mg}$$

Two contributions of $2mg$ from the geometry (gravity reverses its role between the two points) and $4mg$ from the speed difference, and neither depends on the speed. This is a useful check on any vertical-circle answer.

7. *A projectile is launched with speed v_0 from the bottom of a hill whose surface is a straight slope rising at angle ϕ. The launch angle is θ, measured from the horizontal.*

(a) *Find the range R measured along the slope.*

Use ordinary horizontal and vertical axes and impose that the landing point lies on the line $y = x \tan\phi$. With $x = v_0 \cos\theta\ t$ and $y = v_0 \sin\theta\ t - \frac{1}{2}gt^2$,

$$v_0 \sin\theta\ t - \frac{1}{2}gt^2 = v_0 \cos\theta\ t \tan\phi$$

Discarding $t = 0$ and solving,

$$t = \frac{2v_0}{g}\left(\sin\theta - \cos\theta \tan\phi\right)$$

The distance along the slope is $R = \frac{x}{\cos\phi} = \frac{v_0 \cos\theta\ t}{\cos\phi}$, so

$$\boxed{R = \frac{2v_0^2 \cos\theta \sin(\theta - \phi)}{g\cos^2\phi}}$$

where the last step used $\cos\theta\,(\sin\theta - \cos\theta\tan\phi) = \frac{\cos\theta\sin(\theta-\phi)}{\cos\phi}$.

Check $\phi = 0$: this reduces to $\frac{v_0^2 \sin 2\theta}{g}$, the flat-ground range.

(b) *Find the launch angle that maximizes R, and the maximum range itself. Interpret the angle geometrically.*

Rather than differentiate the product, convert it to a single sine using $2\cos A\sin B = \sin(A+B) - \sin(A-B)$:

$$R = \frac{v_0^2}{g\cos^2\phi}\left[\sin(2\theta - \phi) - \sin\phi\right]$$

Only the first term depends on θ, and it is largest when $\sin(2\theta - \phi) = 1$:

$$2\theta - \phi = \frac{\pi}{2} \qquad \Longrightarrow \qquad \boxed{\theta = \frac{\pi}{4} + \frac{\phi}{2}}$$

Substituting back and using $\cos^2\phi = 1 - \sin^2\phi$,

$$R_{max} = \frac{v_0^2(1 - \sin\phi)}{g\cos^2\phi} = \boxed{\frac{v_0^2}{g(1+\sin\phi)}}$$

Geometrically: $\frac{\pi}{4} + \frac{\phi}{2}$ is the angle exactly halfway between the slope (at ϕ) and the vertical (at $\frac{\pi}{2}$). So the rule is always "bisect the angle between the target surface and straight up," which on flat ground gives the familiar 45° and remains true for a downhill slope with $\phi < 0$. Worth remembering: it is far easier to recall than the formula.

Check $\phi \to \frac{\pi}{2}$: $\theta \to \frac{\pi}{2}$ and $R_{max} \to \frac{v_0^2}{2g}$, which is just the straight-up rise height. Correct.

8. *A uniform chain of mass M and length L lies on a frictionless table with a length x_0 hanging over the edge. It is released from rest. Take x to be the length hanging over the edge, and assume the chain stays taut and the corner does not impede it.*

(a) *Show that $\ddot{x} = \frac{g}{L}x$, and comment on how this compares to the mass-on-a-spring equation.*

The whole chain, mass M, moves together with speed $\dot{x}$. The only unbalanced external force is gravity acting on the hanging portion, whose mass is $\frac{M}{L}x$ (the table supports the rest, and the table is frictionless):

$$M\ddot{x} = \frac{M}{L}xg \qquad \Longrightarrow \qquad \boxed{\ddot{x} = \frac{g}{L}x}$$

This is the spring equation $\ddot{x} = -\omega^2 x$ with the sign flipped. That single sign is the entire difference between oscillation and runaway: a spring pulls back toward equilibrium, while here the further the chain slips the harder it is pulled. Accordingly the solutions are cosh and sinh instead of cos and sin, and $x = 0$ is an unstable equilibrium rather than a stable one — a chain balanced with nothing hanging over will sit there forever.

(b) *Solve for $x(t)$ with the given initial conditions.*

With $\omega = \sqrt{\frac{g}{L}}$ the general solution is $x = A\cosh(\omega t) + B\sinh(\omega t)$. Then $x(0) = A = x_0$ and $\dot{x}(0) = B\omega = 0$, so

$$\boxed{x(t) = x_0 \cosh\left(\sqrt{\frac{g}{L}}\ t\right)}$$

(c) *Find the time at which the chain completely leaves the table.*

Set $x = L$:

$$\boxed{t = \sqrt{\frac{L}{g}}\,\cosh^{-1}\left(\frac{L}{x_0}\right) = \sqrt{\frac{L}{g}}\,\ln\left(\frac{L + \sqrt{L^2 - x_0^2}}{x_0}\right)}$$

As $x_0 \to 0$ this diverges logarithmically — slowly. Halving the initial overhang does not double the time; it adds only about $\sqrt{\frac{L}{g}}\ln 2$ to it. Unstable equilibria are left reluctantly but not indefinitely.

(d) *Find the speed of the chain at that moment, and check your answer by an independent method.*

Differentiating, $\dot{x} = x_0\omega\sinh(\omega t)$, and at the moment of departure $\cosh(\omega t) = \frac{L}{x_0}$, so $\sinh(\omega t) = \sqrt{\frac{L^2}{x_0^2} - 1}$. Thus

$$\boxed{v = \sqrt{\frac{g}{L}\left(L^2 - x_0^2\right)}}$$

Check by energy. The hanging piece of length x has its center of mass $\frac{x}{2}$ below the table, so $U(x) = -\frac{M}{L}xg\frac{x}{2} = -\frac{Mgx^2}{2L}$. Then

$$\frac{1}{2}Mv^2 = U(x_0) - U(L) = \frac{MgL}{2} - \frac{Mgx_0^2}{2L}$$

which gives the same v. Energy is available here because the table is frictionless and the chain is inextensible; note that it would *not* be available in the common variant where the chain is piled up and picked up link by link, since that version has inelastic losses.

9. *A block is launched up a rough incline of angle θ with initial speed v_0. The coefficient of kinetic friction is μ_k. Three students propose expressions for the distance d measured along the incline that the block travels before coming momentarily to rest:*

- Student 1: $d = \dfrac{v_0^2}{2g(\sin\theta + \mu_k\cos\theta)}$
- Student 2: $d = \dfrac{v_0^2}{2g(\sin\theta - \mu_k\cos\theta)}$
- Student 3: $d = \dfrac{\mu_k v_0^2}{2g\sin\theta\cos\theta}$

(a) *Without solving the problem, determine which of these can be rejected, and give the specific reason in each case.*

Student 2 can be rejected. Its d *increases* with μ_k and diverges at $\mu_k = \tan\theta$. Friction can only take energy away, so more friction must always mean a shorter distance. Any expression that gets more generous as you make the surface rougher is wrong on inspection.

Student 3 can be rejected. Setting $\mu_k \to 0$ gives $d \to 0$, claiming that a block launched onto a *frictionless* incline stops instantly. Friction is not what stops the block — gravity is — so $\mu_k = 0$ must leave a perfectly good finite answer behind. It also fails at $\theta \to 0$, where it gives $d \to \infty$ despite friction still acting on a horizontal surface.

Student 1 survives every check. It decreases with μ_k; at $\mu_k = 0$ it reduces to $\frac{v_0^2}{2g\sin\theta}$, the frictionless answer; it diverges only when both $\theta \to 0$ and $\mu_k \to 0$, i.e. when nothing at all opposes the motion; and it has the right units.

(b) *Now derive the correct expression.*

On the way up, gravity's component along the incline and friction both point down the incline, so with $N = mg\cos\theta$,

$$ma = -mg\sin\theta - \mu_k mg\cos\theta \qquad \Longrightarrow \qquad a = -g\left(\sin\theta + \mu_k\cos\theta\right)$$

The acceleration is constant, so $v_0^2 = 2\,|a|\,d$ gives

$$\boxed{d = \frac{v_0^2}{2g\left(\sin\theta + \mu_k\cos\theta\right)}}$$

confirming Student 1.

(c) *Does the block slide back down? State the condition, and explain why it involves a different coefficient than the one in part (b).*

At the instant it stops, the block slides back only if gravity along the incline exceeds the maximum available static friction:

$$mg\sin\theta > \mu_s mg\cos\theta \qquad \Longrightarrow \qquad \boxed{\tan\theta > \mu_s}$$

The coefficient is μ_s, not μ_k, because the block is momentarily at rest, and static friction is what governs whether motion begins. Since $\mu_s > \mu_k$ in general, there is a genuine range of angles for which a block will slide up, stop, and stay stopped — which is not something the kinetic-friction analysis of part (b) could ever have told you.