Energy Homework, Part II (Solutions)

Full Name, Period: **due: x/xx**

1. A particle moves in the plane under the force $\vec{F} = k\,(y\hat{x} - x\hat{y})$.

   (a) *Find the work done in moving the particle from the origin to the point (a,a) along the path that goes first along the x axis to $(a,0)$, then straight up to (a,a).*

   On the first leg $y=0$, so $F_x = ky = 0$ and the leg contributes nothing. On the second leg $x=a$ is fixed and we move in y, so only $F_y = -kx = -ka$ matters:

   $$W = \int_0^a (-ka)\,dy = \boxed{-ka^2}$$

   (b) *Now find the work along the path that goes first up the y axis to $(0,a)$, then straight across to (a,a).*

   Same structure, opposite outcome. On the first leg $x=0$ so $F_y = 0$ and nothing happens. On the second leg $y=a$ is fixed and $F_x = ky = ka$:

   $$W = \int_0^a ka\;dx = \boxed{+ka^2}$$

   (c) *Could you define a potential energy for this field? If so, define it. If not, explain why it isn't possible.*

   No, it is not possible. A potential energy has to be a function of position alone, so $U(a,a)$ must be a single number. But the work done getting to (a,a) from the origin was $-ka^2$ along one path and $+ka^2$ along another, so the energy the particle has on arrival depends on how it got there and not merely on where it is. There is no consistent value to assign.

   Said another way: carry the particle out along the first path and back along the second, and the force has done $-ka^2 - \left(+ka^2\right) = -2ka^2$ of work on a trip that ends exactly where it began. Run the loop the other way and you extract $2ka^2$ per lap, as many times as you like. A potential energy is exactly the bookkeeping that rules this out, so a force permitting it cannot have one.

   A correct answer must say either that U would be ambiguous, or that a closed loop returns nonzero work. Asserting "it is not conservative" with no reason attached is not enough.

2. A particle of mass m moves along the x axis in the potential energy

   $$U(x) = U_0\left[\left(\frac{x}{a}\right)^4 - 4\left(\frac{x}{a}\right)^2\right]$$

   where U_0 and a are positive constants.

   (a) *Find all the equilibrium positions and state which are stable and which are unstable.*

   Equilibrium means zero force, so zero slope. Writing $u = \frac{x}{a}$ to keep things clean,

   $$\frac{dU}{dx} = \frac{U_0}{a}\left[4u^3 - 8u\right] = \frac{4U_0}{a}u\left(u^2-2\right) = 0$$

   $$\boxed{x = 0, \quad x = \pm\sqrt{2}\,a}$$

   The energies there are $U(0) = 0$ and $U\left(\pm\sqrt{2}a\right) = U_0\left[4-8\right] = -4U_0$.
   So $x = \pm\sqrt{2}a$ are minima and therefore **stable**, while $x=0$ is a local maximum sitting between them and is **unstable**. This is a double well: two valleys with a hill between them.

   Checking the sign of U'' works too, but sketching the curve and reading it off is faster and less error prone. Either counts.

(b) *The particle is released from rest at $x = a$. Describe the resulting motion, and find the positions where it momentarily comes to rest.*

Released from rest means the total energy is just the potential energy there:

$$E = U(a) = U_0[1 - 4] = -3U_0$$

Since $E < 0 = U(0)$, the particle cannot climb the central hill. It is trapped in the right-hand well, sliding back and forth.
It comes to rest wherever $U(x) = E$:

$$U_0\left[u^4 - 4u^2\right] = -3U_0 \quad \Longrightarrow \quad u^4 - 4u^2 + 3 = 0 \quad \Longrightarrow \quad \left(u^2 - 1\right)\left(u^2 - 3\right) = 0$$

The roots are $u = \pm 1$ and $u = \pm\sqrt{3}$, and only the positive ones lie in the right well:

$$\boxed{x = a \quad \text{and} \quad x = \sqrt{3}\,a}$$

It oscillates between these, moving fastest as it passes the bottom of the well at $x = \sqrt{2}a$.
The negative roots solve the equation perfectly well and describe the identical motion in the left well, which this particle never reaches. A student who lists all four and says which are relevant has answered correctly; one who lists all four with no comment has not.

(c) *Find the minimum kinetic energy the particle would need, at the bottom of the right-hand well, to reach the left-hand well.*

To cross it must clear the hill at $x = 0$, where $U = 0$. Starting at the bottom of the well where $U = -4U_0$,

$$K_{min} + (-4U_0) = 0$$

$$\boxed{K_{min} = 4U_0}$$

Given exactly this much, it arrives at the top with zero speed and stalls there, since the top is an unstable equilibrium. Any more and it crosses.

(d) *Can the particle escape to $x = \infty$ if given enough energy? Explain.*

$\boxed{\text{No.}}$ As x grows the u^4 term takes over and $U \to +\infty$, so there is always more hill ahead no matter how much energy the particle starts with. It is bound at every finite energy.
Compare the gas cloud on the quiz, which was also bound at every energy: there the potential rose only logarithmically and still won. What matters is whether U has a finite ceiling, not how quickly it climbs.

3. A uniform chain of total mass M and total length L lies on a frictionless table, with a length ℓ hanging over the edge. The chain has linear density $\lambda = \frac{M}{L}$.

   (a) *Find the work you must do to pull the hanging part back onto the table, by integrating the force you apply.*

   The point of the problem is that the force needed changes as you pull. When a length x is still hanging, the weight you are supporting is $\lambda x g$, and since the table is frictionless that is the entire force required:

$$W = \int_0^\ell \lambda g x \, dx = \frac{\lambda g \ell^2}{2}$$

$$\boxed{W = \frac{M g \ell^2}{2L}}$$

   (b) *Now find the same work using the center of mass of the hanging piece, and confirm that the two answers agree.*

   The hanging piece has mass $\lambda\ell$, and because it is uniform its center of mass sits a distance $\frac{\ell}{2}$ below the table top. Raising that mass to table level takes

$$W = (\lambda\ell)\, g \frac{\ell}{2} = \frac{\lambda g \ell^2}{2} = \frac{M g \ell^2}{2L}$$

   which matches part (a).
   Both methods work because gravity is conservative: the total work depends only on how far each piece of chain moved, so we may either track the pieces one at a time or replace the whole hanging segment by its center of mass.

   The characteristic error is $W = (\lambda\ell)\, g\ell$, using the full hanging length as the height. Only the bottom link moves that far, the link at the table edge does not move at all, and the average is $\frac{\ell}{2}$.

   (c) *The chain is released from rest with the length ℓ hanging. Find its speed at the moment the last link leaves the table.*

   Use the center of mass at both ends, since only the change in potential energy matters. At the start the hanging length ℓ has its center of mass $\frac{\ell}{2}$ below the table and the rest of the chain is at table level, so

$$U_i = -\lambda \ell g \frac{\ell}{2} = -\frac{M g \ell^2}{2L}$$

   At the end the whole chain hangs with its top at the table edge, so all of M sits $\frac{L}{2}$ below:

$$U_f = -\frac{M g L}{2}$$

   Every link moves at the same speed, since the chain cannot stretch, so the kinetic energy is simply $\frac{1}{2} M v^2$:

$$\frac{1}{2} M v^2 = U_i - U_f = \frac{M g L}{2} - \frac{M g \ell^2}{2L}$$

$$\boxed{v = \sqrt{\frac{g\,(L^2 - \ell^2)}{L}}}$$

   Check the limits. Setting $\ell = L$ gives $v = 0$, correct, since then the chain was already fully hanging and never moved. Very small ℓ approaches $\sqrt{gL}$, the fastest this chain can possibly be going.
   The frictionless table is doing real work in this problem. With friction, or with a chain that piles up and jerks as it goes over, energy is lost and this answer becomes an overestimate.

4. A car of mass m starts from rest and its engine delivers a constant power P. Ignore all friction and drag.

(a) *Find the speed as a function of time.*

Constant power means the kinetic energy grows linearly in time, since $P = \frac{dK}{dt}$:

$$\frac{1}{2}mv^2 = Pt$$

$$\boxed{v(t) = \sqrt{\frac{2Pt}{m}}}$$

This is an energy argument from start to finish. Writing $F = ma$ first leads to a differential equation and considerably more work.

(b) *Find the distance traveled as a function of time.*

$$x(t) = \int_0^t \sqrt{\frac{2P\tau}{m}}\, d\tau = \sqrt{\frac{2P}{m}} \cdot \frac{2}{3} t^{3/2}$$

$$\boxed{x(t) = \frac{2}{3}\sqrt{\frac{2P}{m}}\, t^{3/2}}$$

5. A planet of mass M and radius R has uniform density. A narrow tunnel is drilled straight through its center, and a ball of mass m is dropped from rest at the surface. Neglect all friction and air resistance.

(a) *Show that the gravitational force on the ball while it is inside the planet is $\vec{F} = -\frac{GMm}{R^3} r\,\hat{\mathrm{r}}$, and use this to find the potential energy inside.*

Only the mass at radius less than r pulls on the ball. With uniform density that enclosed mass is

$$M(r) = M\frac{r^3}{R^3}$$

so

$$\vec{F} = -\frac{GM(r)m}{r^2}\hat{\mathrm{r}} = -\frac{GMm}{R^3} r\,\hat{\mathrm{r}}$$

The force grows *linearly* from the center rather than blowing up: as you burrow inward, the shells you leave behind stop pulling on you.

To get the potential energy, integrate inward from the surface, where $U(R) = -\frac{GMm}{R}$:

$$U(r) = U(R) - \int_r^R \frac{GMm}{R^3} r'\, dr' = -\frac{GMm}{R} - \frac{GMm}{2R^3}\left(R^2 - r^2\right)$$

$$\boxed{U(r) = -\frac{GMm}{2R^3}\left(3R^2 - r^2\right)}$$

Check: at $r = R$ this gives $-\frac{GMm}{R}$, matching the formula for outside the planet.

(b) *Find the speed of the ball as it passes through the center.*

$$\frac{1}{2}mv^2 = U(R) - U(0) = -\frac{GMm}{R} + \frac{3GMm}{2R} = \frac{GMm}{2R}$$

$$\boxed{v = \sqrt{\frac{GM}{R}}}$$

Note this is $\frac{1}{\sqrt{2}}$ times the escape speed from the surface, and equal to the speed of a satellite skimming the surface.

The ball does not stop there. It continues to the far side, arriving with zero speed, and comes back. What happens next is a question for the oscillator unit.