Physics Notes

Contents

Chapter 1

Introduction

1.1 Statement of Purpose and Credits

These notes have been a number of years in the making. I sincerely hope that you find them useful. They are meant to be a quick repository for the most important content in the course. They are not meant as a substitute for class notes. If you find any typos, or think that something is unclear, please let me know so that I can address it. I am also working on a repository for additional resources that students might find useful. If you know of any good resources, please let me know. You are welcome to distribute the materials to other students, and to friends, but please do not post them in any public forum.

1.2 Useful Outside Materials

- Textbooks: if you find a good textbook at this level, please let me know. I don't know any.
- Simulations: I strongly recommend looking into a simulation if you are struggling to grasp something. The ability to visualize can really help.
  - `ophysics.com`: excellent resource for much of what is covered here and in later courses
  - `phet.colorado.edu`: has a few good resources, notably on resonance. The level of the material varies from middle school to second year of college.
- Practice problems
  - 5 steps to a 5 is a decent study guide for AP 1 and 2. It is primarily a repository for practice problems. I personally don't find the brief conceptual discussions very enlightening, but they aren't incorrect.
  - `https://www.crackap.com/ap/physics-1/`: a lot of practice problems for the AP one exam. I obviously did not go through all of it, but what I looked at was good. Might be *slightly* easier than the AP.
- Conceptual Discussion: This is the purpose of these notes. I don't know any places that discuss the concepts well aimed at the AP level.

1.3 Image Credits

All of the figures are my own creations, but I would like to call out some of the free software that really helped me.

1. Many of the figures in here were created with Geogebra, in particular everything in the prerequisites section and the torque section, and most of the oscillations and waves chapter. It is a really cool piece of software that can be found at https://www.geogebra.org/. If you struggle with Geometry, I can't recommend it more highly.

2. Graphs in the paper were made with either Geogebra, Graph (available at https://www.padowan.dk/download/ *), or various programming languages.

3. The pulley systems were mostly made in Tikz. I really can't recommend this software. It's use made sense when these figures were made (mostly in undergrad and graduate school), but much better alternatives exist now for most purposes.

*use the sourceforge link from there if the download doesn't work directly from the site

Chapter 2

Mathematical Prerequisites

In this chapter some of the mathematical structures that we will use this year will be laid out. You don't necessarily need to be an expert in them immediately since we will refer back to them a number of times over the course of the year, but you should be fluent enough with them to apply them to simple problems.

2.1 Vectors

2.1.1 Definition of vector

Vectors are typically defined mathematically in terms of the mathematical rules that they obey. Most undergraduate linear algebra texts provide rules for vector spaces, the elements of a vector space are vectors by definition. An easier way to think about them, which will be good enough for a freshman physics class* is as quantities with both magnitude and direction. Basically, if it makes sense to represent something with an arrow (velocity, displacement, force...) it's a vector. If it doesn't make sense (temperature, energy...) its a scalar.

2.1.2 Notation for vectors

Many notations exist for vectors. Here I will use an arrow over the top of a variable. For example velocity is denoted $\vec{v}$. I will endeavor to be very careful to distinguish vectors from scalars. Generally if a variable that is usually a vector is written without the arrow, we are referring to the magnitude of that variable. For example v represents the magnitude of the velocity, which is an object's speed. Be careful when interpreting this; the magnitude of displacement is not the distance traveled by the object, for example.

2.1.3 Components of vectors

When we want to write out a vector, we will typically do so in terms of its Cartesian representation. We use the variables $\hat{\mathrm{x}}$,$\hat{\mathrm{y}}$ and $\hat{\mathrm{z}}$† to represent the three directions an object can travel. Technically each is a vector with length one that points in the indicated direction. These are called unit vectors, but you can think of them as mathematical representations of directions. $\hat{x}$ is only used for horizontal, while $\hat{z}$ is only used for vertical. $\hat{y}$ can be used for either depending on context.

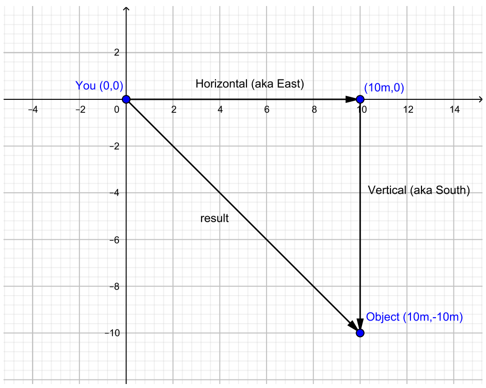
Interpretation of vector components

The way to think about vector components is that we are breaking up an arbitrary vector $\vec{w}$ into a set of perpendicular vectors that when added give the total vector. This is still a bit abstract, so let's imagine

*If you plan to ultimately take quantum, you should learn and understand the real definition

†pronounced x-hat, y-hat, z-hat. $\hat{\imath}$,$\hat{\jmath}$ and $\hat{k}$ are also used in some textbooks

that we are pushing on a block with a force $\vec{F} = 5\ \text{N}\hat{x} + 10\ \text{N}\hat{y}$. What this means is that the force $\vec{F}$ being applied to the block produces the exact same result as two forces $\vec{F_1} = 5\ \text{N}\hat{x}$ and $\vec{F_1} = 10\ \text{N}\hat{y}$. The same works for displacement, and the result is even easier to visualize. For this example, let's let $\hat{x}$ represent East and $\hat{y}$ represent North. If I tell you that an object on the floor has a position of $\vec{r} = 10\ \text{m}\hat{x} - 10\text{m}\hat{y}$ relative to you, this means you could get to the object by walking 10 m East and then 10 meters South as shown in the diagram below.



Sine/Cosine and vector components

We already have functions that can gives us the components of a vector if we only have a magnitude and angle. You may have heard the cosine function defined in terms of triangles as $\cos\theta = \frac{adj}{hyp}$. This can be rearranged to give $adj = (hyp)\cos\theta$. Let's now create a triangle with a side along the x axis and a completely vertical side as shown in figure 2.1. Notice that the length of the vertical side is given by $r\sin\theta$ and the horizontal is given by $r\cos\theta$. If we apply this to a vector $\vec{r}$ (with magnitude r), then $r\cos\theta$ gives the length of the side that points along the $\hat{x}$ direction, which is the magnitude of the x component of the vector. In the figure, $\vec{r}$ has length 5 and makes an angle of $\theta \approx 36.87°$. Then the x side has length $5\cos(36.87°) = 4$ and the y side has length $5\sin(36.87°) = 3$. This fits with our other representation of the vector as $\vec{r} = 4\hat{x} + 3\hat{y}$.

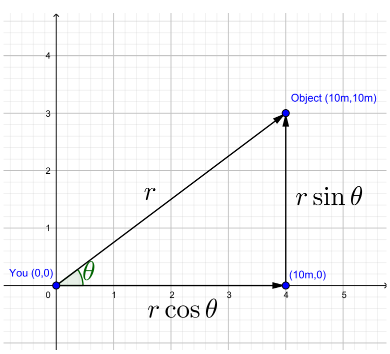


Figure 2.1: The magnitude multiplied by the cosine of the angle gives the length of the side that points along $\hat{x}$ while the magnitude multiplied by the sine of the angle gives the length of the side that points along $\hat{y}$.

This method works for any vector. If we want to know the length of the $\hat{x}$ and $\hat{y}$ components of a vector, we can just take $w\cos\theta$ and $w\sin\theta$ respectively to get them. I like to think of this as a geometric projection effect. If I took a flashlight and shined it from directly above, the length of the shadow that would appear on the x axis would be given by the length of the x component of the vector. If I shined it from the right, the length of the shadow on the y axis would be given by the y component of the vector. This intuition will be useful when we talk about dot product in section 2.1.5.

2.1.4 Adding Vectors

When we want to add two vectors, we simply add up the components of the vectors, pretending that the directions are variables like any other.

Example: Find the net force when forces of $F_1 = 15\ \mathrm{N}\hat{\mathrm{x}} - 10\ \mathrm{N}\hat{\mathrm{y}} + 4\ \mathrm{N}\hat{\mathrm{z}}$ *and* $F_2 = -5\ \mathrm{N}\hat{\mathrm{x}} + 9\ \mathrm{N}\hat{\mathrm{y}} - 3\ \mathrm{N}\hat{\mathrm{z}}$ *are simultaneously applied to the same object.*

$$\vec{F}_{net} = \vec{F}_1 + \vec{F}_2$$

Now we just add up each direction separately. Plugging in the components gives

$$\vec{F}_{net} = (15 - 5)\ \mathrm{N}\hat{\mathrm{x}} + (-10 + 9)\ \mathrm{N}\hat{\mathrm{y}} + (4 - 3)\ \mathrm{N}\hat{\mathrm{z}}$$

$$\boxed{\vec{F}_{net} = 10\ \mathrm{N}\hat{\mathrm{x}} - 1\ \mathrm{N}\hat{\mathrm{y}} + 1\ \mathrm{N}\hat{\mathrm{z}}}$$

2.1.5 The dot product

We would like a method to determine how close to being parallel two vectors are. In section 2.1.3 we used the cosine function to find components of a vector, basically, we showed that it gives the x component of a vector when the triangle is aligned appropriately. Put another way, the cosine gave us an idea of how close a vector was to being parallel with the x axis (equivalently the length of the shadow cast on the x axis). If you don't see that immediately, think about how $r\cos\theta$ behaves for different θ. If $\theta = 0$ then the vector points completely along x, so $r\cos 0° = r$ and if $\theta = 90°$, then the vector is along y, which is completely perpendicular to x, and $r\cos 90° = 0$. There was not any strong reason that the sides had to be aligned along the axes, so let's imagine a new case where they are not. Cosine should serve the same function that it did before! The cosine of the angle between the vectors should tell us how close they are to being parallel. An example of non-parallel vectors is in figure 2.2.

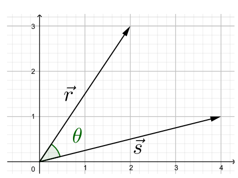


Figure 2.2: Two vectors $\vec{r}$ and $\vec{s}$ that are separated by an angle θ. As θ gets larger, the vectors become nearer to perpendicular.

Now we use this intuition to define the dot product. The dot product is the length of the first vector times the length of the second (our intuition for multiplication) times the cosine of the angle between them to compensate for the geometric effect. Where $||\vec{w}||$ is the magnitude $\vec{w}$ (also denoted with just w in physics) We denote the dot product like this:

$$\boxed{\vec{r} \cdot \vec{s} = ||r||||s|| \cos\theta}$$

Geometrically, this is the length of the shadow that the smaller vector would cast on the longer one multiplied by the length of the longer one.
If we have both vectors in terms of components, the dot product is easy to calculate; simply take the vector components and multiply them together, remembering that we included the angle between the vectors. This means that for components that would multiply $\hat{\mathrm{x}} \cdot \hat{\mathrm{y}}$ the value will always be 0 because the cosine between

those directions is 90°.

Example: Calculate $\vec{w} \cdot \vec{s}$ *if* $\vec{w} = 5\hat{x} + 4\hat{y} + 2\hat{z}$ *and* $\vec{s} = 7\hat{x} + 6\hat{y} + 3\hat{z}$

$$\vec{w} \cdot \vec{s} = (5\hat{x} + 4\hat{y} + 2\hat{z}) \cdot (7\hat{x} + 6\hat{y} + 3\hat{z})$$

If we were doing normal multiplication, we would distribute everything out, but this would involve terms like $(5\hat{x}) \cdot (4\hat{y}) = 20\hat{x} \cdot \hat{y}$ which is 0 because $\hat{x}$ and $\hat{y}$ are perpendicular.
The only terms we need to include are ones with just $\hat{x}$, just $\hat{y}$, or just $\hat{z}$.

$$\vec{w} \cdot \vec{s} = (5\hat{x}) \cdot (7\hat{x}) + (4\hat{y}) \cdot (6\hat{y}) + (2\hat{z}) \cdot (3\hat{z})$$

Using that the dot product is associative

$$\vec{w} \cdot \vec{s} = 35\hat{x} \cdot \hat{x} + 24\hat{y} \cdot \hat{y} + 6\hat{z} \cdot \hat{z}$$

Now we note that every vector is exactly perpendicular to itself. Since any unit vector has length 1 by definition $\hat{x} \cdot \hat{x} = 1$ and the same with $\hat{y}$ and $\hat{z}$.

$$\vec{w} \cdot \vec{s} = 35 + 24 + 6$$
$$\boxed{\vec{w} \cdot \vec{s} = 65}$$

2.1.6 Magnitude of a vector

Fortunately, now that we have the dot product, finding the magnitude is easy. Since every vector is parallel to itself $\vec{v} \cdot \vec{v} = v^2$ so to get the magnitude, we just use

$$\boxed{v = \sqrt{\vec{v} \cdot \vec{v}}}$$

2.2 Linearization

2.2.1 Goal

We want to be able to take a complicated variable dependence and produce a simple graph of the form $y = mx + b$ for analysis.

2.2.2 Process

Lets say we have an expression that looks like $z^m = abcw^n$ where a is some constant whose value we want to measure experimentally; z and w are variables; b and c are other constants that we know already. If we plot this, it will be non-linear and difficult to interpret. Instead, let's try to make it look like a line. We can do this be defining new variables. In our case

1. We don't like z^m, so let's define

$$\boxed{y = z^m}$$

2. We don't like bcw^n, so let's define

$$\boxed{x = bcw^n}$$

(note, we can't get rid of a this way, or there would be nothing to solve!)

Now our original equation ($z^m = abcw^n$) can be written as $y = ax$, which is linear and easy to analyze. The constant we were trying to solve for, **a is now the slope of the graph of y vs x.** So we can easily find the value of a just by graphing. Let's look at a specific example with context.

2.2.3 Example: Force of gravity

The force of gravity between two objects is $F = \frac{GM_1M_2}{R^2}$
We measure the force on 2 objects $M_1 = \frac{1}{6.67} \times 10^{11}$ kg and some unknown mass M_2 at different distances and get the following chart:

Force on objects as distance is varied

R (m)	F (N)
1	100
2	25
3	11.1

Clearly plotting this is going to be obnoxious since the data spans an entire order of magnitude. If you did, you would get something nonlinear.

The goal is to make something that looks like $y = mx + b$ out of this. F is fine right away, so we just assign $y = F$. The other side is harder. We have $y = M_2\frac{GM_1}{R^2}$. The best bet is to just use $x = \frac{GM_1}{R^2}$.

Now we add a new column to our chart and calculate the values for that chart from what we already know. You will be given any needed constants for labs and exams, in this case $G = 6.67 \times 10^{-11}\ \frac{\mathrm{N \cdot m^2}}{(\mathrm{kg})^2}$

Force on objects as distance is varied

R (m)	F (N)	x=$\frac{GM_1}{R^2}$ ($\frac{N}{\mathrm{kg}}$)
1	100	1
2	25	.25
100	11.1	.11

Normally we would plot this data and use a trend line to find the slope, but in this case it is fairly clear that the slope is 100 kg so

$$\boxed{M_2 = 100 \text{ kg}}$$

2.3 Dimensional Analysis

2.3.1 Goal

We want to use the units of physical quantities to derive an equation that relates those quantities.

2.3.2 Process

Let's say you want to derive an expression for z in terms of w, x, and y

1. Write out the units of what you want at the end (z in this case)
2. Write out the units of the quantities you have (w, x, y in this case)
3. Rewrite the units of what you want (z) so that no unit appears there that does not appear in what you have (w, x, y)
4. Find which of the quantities you have (w,x, or y) has a unique unit in. Start with that one. Let's assume w was our choice.

   If no quantity has any unique unit or multiple do, use the one with the most complicated units.
5. Multiply by one in order to get all of the units of w (to any power) to appear in the units of z.
6. Replace the units of w in z with $[w]$.
7. Repeat steps 4-7 with all remaining variables.

2.3.3 Example: Drag Force

Drag forces result when an object is moving through a fluid. It is reasonable to assume that the drag force F ($[F] = \mathrm{N}$) might depend on the density of the fluid ρ ($[\rho] = \frac{\mathrm{kg}}{\mathrm{m}^3}$) we are moving through (objects slow down faster in water than air), the velocity of the object v ($[v] = \frac{\mathrm{m}}{\mathrm{s}}$), and the cross sectional area of the front of the object A ($[A] = \mathrm{m}^2$). Use dimensional analysis to find an approximate formula for F in terms of the other variables given. The steps here are labeled according to the steps in the process above.

1. Write out the units of what we want, in this case
$$[F] = \mathrm{N}$$

2. Write out the units of everything we have, in this case
$$[\rho] = \frac{\mathrm{kg}}{\mathrm{m}^3}$$
$$[v] = \frac{\mathrm{m}}{\mathrm{s}}$$
$$[A] = \mathrm{m}^2$$

3. We don't have N anywhere in the other variables, so we need to rewrite $[F]$ (using $F = ma$)
$$[F] = kg \cdot \frac{\mathrm{m}}{\mathrm{s}^2}$$
We now want to separate $[F]$ into clumps that match the units of what we have.

4. We start with something that has a unique unit. In this case kg comes only from ρ

5. We multiply $[F]$ by $1 = \frac{\mathrm{m}^3}{\mathrm{m}^3}$ to make $[F]$ look like $[\rho]$.
$$[F] = \frac{\mathrm{kg}}{\mathrm{m}^3} \cdot \mathrm{m}^3 \cdot \frac{\mathrm{m}}{\mathrm{s}^2}$$

6. We pull out $[\rho] = \frac{\mathrm{kg}}{\mathrm{m}^3}$ from $[F]$ to get
$$[F] = [\rho] \cdot \mathrm{m}^3 \cdot \frac{\mathrm{m}}{\mathrm{s}^2}$$

7. Now we are done with $[\rho]$ so we go back to step 4.

4. We see that $[F]$ has s^2 in the bottom, and seconds only appear in velocity.

5. No multiplication is necessary because we already have plenty of meters. So we just rewrite $[F]$ to look like $[v^2]$
$$[F] = [\rho] \cdot \frac{\mathrm{m}^2}{\mathrm{s}^2} \cdot \mathrm{m}^2$$

6. We replace $\frac{\mathrm{m}^2}{\mathrm{s}^2}$ with $[v^2]$.
$$[F] = [\rho][v^2]\mathrm{m}^2$$
We could go through the steps again, but no rewriting is necessary because $\mathrm{m}^2 = [\mathrm{A}]$ So our formula is
$$\boxed{F \propto \rho v^2 A}$$

2.4 Trend Lines

2.4.1 Why we need trend lines

When we get data from experiments, our results will never align perfectly with what we expect. This is due to a combination of imperfect instruments, our minor mistakes when taking data, and the impossibility of completely controlling all outside factors that might influence our experiment. Trend lines let us get useful information from messy data.

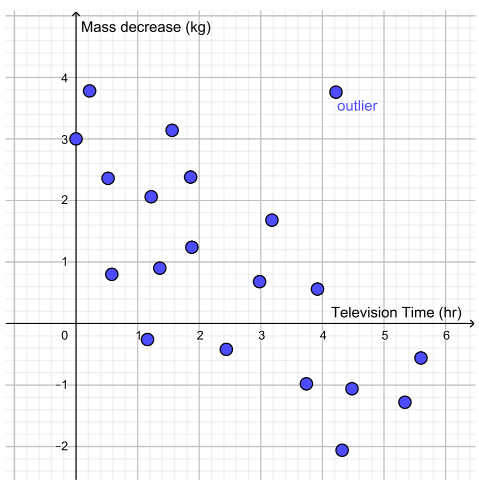
2.4.2 A massive undertaking

The experiment

As an example, imagine that we used a bathroom scale to measure the weight of a number of people who wanted to lose weight. We then measured how many hours of tv each of them watched each week for a year. At the end of the year we measured each person's weight again. Finally we plotted the weight decrease against the number of hours of television watched. Mathematically the weight decrease is given by

$$W_{decrease} = W_{start} - W_{end}$$

The graph is given below. Note that we are plotting the weight decrease, that means that positive values on the y axis mean weight was lost, negative values mean weight was gained. Intuitively we expect that people who watched more tv in the last month would lose less weight (or gain more) than those who watched less television.



First, let's think about some reasons why the data doesn't follow a perfect line. These aren't all the reasons, but enough to get the idea.

1. We aren't controlling for what else people are doing with their time.
2. Our bathroom scale isn't perfectly accurate.
3. We aren't controlling for what people are wearing.

In spite of all these errors, we would still like to be able to come to a conclusion based on the results of our experiment. To do that we will use a trend line.

Dealing with outliers

Notice that one point on our graph is really far from all the others. I labeled it "outlier." If we were just handed this data, we probably would not know what to do with the outlier. Since we are the ones conducting this experiment, we can figure it out. If the outlier was caused by something that we can easily determine we should discard that point before making a trend line. An example here would be if the person was wearing a heavy leather jacket for the first weight measurement, but only a swimsuit for the second. If we can't easily explain the outlier, we should normally keep the point. Let's assume that the outlier was from something obvious so that we can remove it.

Finding a trend

We want to find the line that best fits the data given

1. The line should follow the trend of the data.

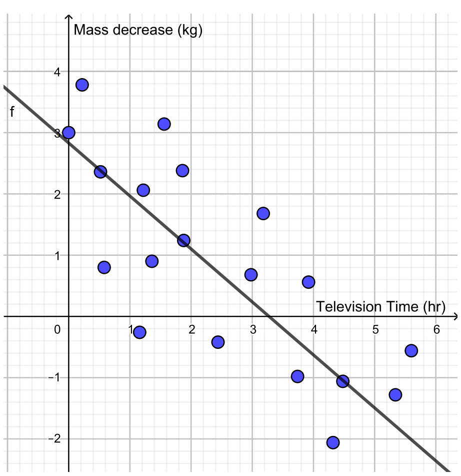
   A good way to check this is look at a few points on the left of the graph and a few points on the right. Are the points on the left higher or lower than the ones on the right? Your trend line must behave the same way as the points.

2. The line should be chosen so that it is close to being in the center of the points.

3. The line does **NOT** need to go through the origin.

4. The trend line does **NOT** need to go through any specific points on the graph.

In our case the trend line looks like this:



Note that I did this trend line by eye, like you will be doing, so it isn't exact. You will not be punished on quizzes, homework, or exams for missing by a little bit, as long as you are close, it's good enough. If we really need high accuracy, there are mathematical methods that can do much better.

Finding the equation of the trend line

Now that we have a line, we want to find the equation. The general equation for a line is

$$y = mx + b$$

Where m is the slope and b is the intercept. To find b we just look at where the graph intersects with the y axis. In our case that is

$$b = 2.8 \text{ kg}$$

To find m we use the equation for slope

$$m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$

and select two points. Technically I can take any two points, but choosing one point to be

$$P_1 = (x_1, y_1) = (0, b)$$

is usually convenient. In our case that is

$$P_1 = (0, 2.8\text{kg})$$

For the second point we can take any point on the graph. I choose to take

$$P_2 = (4\text{hr}, -0.6\text{kg})$$

Plugging these points into our equation for slope gives

$$m = \frac{-0.6\text{kg} - 2.8\text{kg}}{4\text{hr} - 0} = \frac{-3.4}{4}\frac{\text{kg}}{\text{hr}}$$

So our equation is

$$\boxed{y = \frac{-3.4}{4}\frac{\text{kg}}{\text{hr}}x + 2.8 \text{ kg}}$$

Using the trend line

We can then use the trend line to predict values. **(Important note: we need to be very careful when extrapolating outside a data set!)** For example, let's try to predict the weight loss for a new person who watched 4 hours of tv each week. To do that we plug 4 hours in for x.

$$y = \frac{-3.4}{4}\frac{\text{kg}}{\text{hr}}(4\text{hr}) + 2.8\text{kg} = -3.2\text{kg} + 2.8\text{kg} = -0.6\text{kg}$$

The negative sign means mass increased rather than decreased. So our person would be expected to gain 0.6 kg in that year.

Note that we could mathematically plug in something ridiculous like 168 for x. This would correspond to someone spending literally all day watching tv every week for an entire year. Clearly this doesn't make sense physically. Likewise, negative numbers make no physical sense. When we end up with an equation like this we always need to take a second to consider what are reasonable input and output values and avoid using the model where it doesn't apply. Unless you have good reason to think otherwise, you should probably not extrapolate past the last data point most of the time.

Chapter 3

The Conservation Laws

Higher level physics is usually not presented in terms of velocity and mass, but instead in terms of energy and momentum. Technically everything at this level *could* be understood without the conservation laws. It is better both for this course and future courses to think of the conservation laws and symmetries as fundamental.*

Conservation laws essentially state that some quantity will remain constant in the universe. There are many conservation laws in physics (eg mass, charge...) but we will primarily focus on three of them. Energy, momentum, and angular momentum.

3.1 Energy

Unfortunately, there is not a nice definition of energy. One that is often given in textbooks is "Energy is the capacity to do work." This is nice and compact, but also manifestly wrong. In particular, thermal energy is provably unable to do an amount of work equal to the amount of thermal energy present since this violates the second law of thermodynamics. The best that we can actually say is that energy is a mathematical convenience. If you really need an intuitive idea of energy, I like to think of it as the ability to do damage, but this isn't rigorous. Nobel prize winning physicist Richard Feynman explained energy rather poetically:

> There is a fact, or if you wish, a law, governing all natural phenomena that are known to date. There is no known exception to this law - it is exact so far as we know. The law is called the conservation of energy. It states that there is a certain quantity, which we call energy, that does not change in the manifold changes which nature undergoes. That is a most abstract idea, because it is a mathematical principle; it says that there is a numerical quantity which does not change when something happens. It is not a description of a mechanism, or anything concrete; it is just a strange fact that we can calculate some number and when we finish watching nature go through her tricks and calculate the number again, it is the same. (Something like the bishop on a red square, and after a number of moves -details unknown- it is still on some red square. It is a law of this nature.)

What Feynman was getting at is one of the most concise natural laws yet discovered. The total energy of the universe (or any closed system) never changes, the energy just changes form. Mathematically

$$\boxed{\Delta E_{tot} = 0} \tag{3.1}$$

This is among the most important formulas in a Freshman Physics course, and it is the only reason we really care about energy. It's conservation makes our lives easy. Before we can take advantage of this amazing fact, we need to know the types of energy.

*Noether's theorem essentially states (I'm oversimplifying a bit) that symmetry and conservation are the same thing. It is, in my opinion the most interesting result in mathematical physics.

3.1.1 Types of Energy

- Kinetic Energy
  - Energy of motion
  - $E_k = \frac{1}{2}mv^2$
- Potential Energy
  - Energy from position
  - Only differences in potential energy matter, not the actual values.
  - Potential energy is only defined for systems consisting of multiple objects. A single object cannot have potential energy! (This will make more sense after reading the whole handout.)
  - Gravitational Potential Energy
    * Energy resulting from position in a gravity field
    * $E_g = mgh$ near Earth surface
    * $E_g = -\frac{GMm}{R}$ more generally. The negative sign indicates that the system has less energy than when it started.[†]
  - Spring Potential Energy
    * Energy that results from compression or extension of spring
    * $E_s = \frac{1}{2}k(\Delta r)^2$
  - Electrical Potential Energy
    * Energy from position in an electric field
    * $E_e = \frac{kq_1q_1}{R}$ We will explore this in detail later.
- Some other forms
  - Thermal energy
    * Sum of the random kinetic energies of all particles in a system as a result of thermal motion
  - Chemical/Nuclear Potential
    * Energy that comes from chemical or nuclear bonds[‡]
  - Rest mass energy
    * Energy from the existence of matter
    * $E_{rm} = mc^2$

3.1.2 An illustrative example

An object with mass M is placed (at rest) on the top of a frictionless ramp with height H that makes an angle θ from the horizontal then released. Find the velocity of the object when it reaches the ground. Neglect air resistance.

[†] A more rigorous way to think about this is that the gravitational force is always attractive. If we want to separate objects that are gravitationally bound, we have to add energy. Since absolute energy is defined as 0 for a pair of objects infinitely far away, this means that any objects closer than infinity have negative energy from gravity

[‡] This is a bit misleading, the energy isn't stored in the bonds usually, it is stored in the ability to make new tighter bonds

Kinematics: The awful way

Note that I am being as clever as possible with kinematics here and using the solutions to some problems I already know. If I didn't do that, this problem would be several times longer and way harder.
As we normally do, define $\hat{x}$ to be the direction down the ramp and $\hat{y}$ to be the direction perpendicular to the ramp. The acceleration down the ramp is given by

$$a_x = g\sin\theta$$

Then we need to find length of the ramp (Δx) which is the displacement we need to get to the end. We can do that with trig $H = \Delta x \sin\theta$ so

$$\Delta x = \frac{H}{\sin\theta}$$

Then we can use the position equation, with no initial velocity to get

$$\Delta x = \frac{1}{2}a_x t^2$$

Solving for t gives

$$t = \sqrt{\frac{2\Delta x}{a_x}}$$

Now we use the velocity equation with no initial velocity:

$$v_f = a_x t$$

Plugging in our value for t gives

$$v_f = a_x\sqrt{\frac{2\Delta x}{a_x}} = \sqrt{2a_x\Delta x}$$

Plugging in the values for a_x and Δx gives (after some algebra)

$$v_f = \sqrt{2gH}$$

Energy: The easy way

We know that energy will be conserved for the system. Since there is no friction or air resistance, there is no way for energy to be transferred into thermal energy (or other types). This means that all starting potential energy ends up as kinetic energy so that

$$Mgh_i + \frac{1}{2}Mv_i^2 = Mgh_f + \frac{1}{2}Mv_f^2$$

Since we aren't moving at the start $\frac{1}{2}Mv_i^2 = 0$
Since only changes in potential energy matter, let's just define the potential energy to be 0 at ground level. This means that $Mgh_f = 0$ and $Mgh_i = mgH$ so that we have

$$MgH = \frac{1}{2}Mv_f^2$$

The M cancels, and we can solve for v_f to get

$$v_f = \sqrt{2gH}$$

3.1.3 Work

Work can be defined as "the change in the energy of a system as a result of macroscopic interactions with the environment."[§]
This is a complex definition. Note that if thermal energy is ignored (as in a mechanics class and most intro physics courses here) all interactions are macroscopic.

Conservative and Non-Conservative Forces

Before we can refine our definition of work, we need to consider the concept conservative and non-conservative forces.

- A conservative force is one where the work done to move an object between two points is independent of the path taken. This is equivalent to the statement that we can define a potential energy term that depends only on position. Some examples:
  - Gravity is conservative. $E = -\frac{GMm}{R}$ (simplifies to $E = mgh$ near Earth's surface)
  - The electric force is conservative. $E = \frac{kq_1q_2}{R}$
  - For a conservative force, we have the option to include both objects in our system and define the potential energy term, or only include one object and allow the other object to exert an external force. (See section 3.1.6 for an example)
- For non-conservative forces, the work done depends on the path. This usually means we cannot define a potential energy term at all. An example:
  - Friction is non-conservative. If an object travels in a circle with friction and returns to its starting point, its final position is the same as its initial position, but its mechanical energy is different because some of the energy has been converted to thermal energy.

Work for Conservative Forces

For a conservative force, the path taken doesn't matter. This means that we can define work in terms of displacement (i.e. change in position)

$$\boxed{W = \vec{F} \cdot \Delta\vec{r}}\text{¶} \tag{3.2}$$

To understand why the dot product becomes involved, consider the 2-D case of a box sliding on a floor with some velocity $\vec{v} = v_x\hat{\text{x}}$.

- If we pull on the box in the $+\hat{\text{x}}$ direction (ie parallel to the velocity), the velocity increases.
- If we pull in the $-\hat{\text{x}}$ direction (ie in the opposite direction as our velocity), the velocity decreases.
- If we pull in the direction perpendicular to the box's velocity, we have circular motion with no velocity change.

3.1.4 Work Done by Friction

For friction, drag, or any other force that always opposes relative motion, the force always acts to take as much energy as possible out of the system. This means that

$$\boxed{W = -FD} \tag{3.3}$$

where D is distance traveled. In the case of traveling around a circle, displacement is clearly 0, but distance traveled is $D = 2\pi R$, so the work done on the system would just be $W = -2\pi RF$.

[§] This is the definition usually taken in thermodynamics. It is equally applicable to mechanics.

[¶] For formulas involving work, energy is understood to mean non-thermal energy. For our class that means mechanical energy sometimes denoted E_{mech} or TME.

3.1.5 Work and Energy Changes

We said that the energy of a closed system remains constant, but sometimes it isn't convenient to include everything in our system. In this case, the change in energy of the system is the work done on the system. **Important note:** sign conventions for work are different in different textbooks; usually in physics, work done on a system is positive.‖

$$\boxed{W = \Delta E_s} \tag{3.4}$$

Where E_s is the total energy of the system. Note that this reproduces the definition of work that we saw in the previous section. Unless you are specifically given $\vec{F}$ and $\Delta\vec{r}$ the definition of work as the change in energy is easier to use than the definition in terms of $\vec{F}$.

If there are no potential energy terms defined (either because we didn't include the interacting objects in our system, or because there are no conservative forces), then the work done on the system is still it's change in energy, but there is no potential energy, so $\Delta E = \Delta KE$

$$W = \Delta KE_s \tag{3.5}$$

‖This is because physicists are interested in what happens to the system. Other disciplines are more concerned with what the system can do for us, so they take the convention that work done by the system is positive instead

3.1.6 An Example of Choosing a System

Imagine that we have an object at a height h. We drop it and want to know it's velocity when it hits the ground. We can either take just the object to be our system, or the object and the Earth.

Just the object

If we take just the object to be the system we cannot define potential energy** than gravity is an external, conservative force.
We use that $W = \vec{F} \cdot \Delta\vec{r}$. In our case the force is the object's weight $\vec{F} = -mg\hat{z}$ and the displacement is $\Delta\vec{r} = -h\hat{z}$ then

$$W = (-mg\hat{z}) \cdot (-h\hat{z}) = mgh$$

We then use the equation $\Delta E = W$ which simplifies to $W = \Delta KE$ because there is no potential energy term, to get the change in kinetic energy of the system

$$\Delta KE = mgh$$

We plug in the definition $\Delta KE = KE_f - KE_i = \frac{1}{2}m(v_f^2 - v_i^2)$ with $v_i = 0$ since the object was dropped

$$\frac{1}{2}mv_f^2 = mgh$$

or

$$v_f = \sqrt{2gh}$$

The Object and Earth

If we take the Earth as part of the system, potential energy becomes well defined. When we define it, there is now no external force on the system, so $\Delta E = 0$ or

$$E_f = E_i$$

$$E_i = mgh$$

$$E_f = \frac{1}{2}mv_i^2$$

so our equation becomes

$$\frac{1}{2}mv_f^2 = mgh$$

as before we can solve this to get

$$v_f = \sqrt{2gh}$$

**Because we have nothing to define against. Imagine that you are watching the object fall and have no knowledge of the existence of Earth. You can measure that the object is accelerating and infer that there is a force, but you would have no idea how that force would behave.

3.1.7 The "timeless" equation

Here I will derive what some people refer to as the "timeless equation" from kinematics. You shouldn't memorize it because, it is just conservation of energy (oversimplified to the point that it is sometimes wrong).
From conservation of energy in the most general case (ie the case where we are taking only the object as the system, so all forces are external and no potential energies exist), we have

$$\Delta E = W$$

$$\frac{1}{2}Mv_f^2 - \frac{1}{2}mv_i^2 = \vec{F} \cdot \Delta\vec{r}$$

The timeless equation makes the assumption that we are in 1-D so that $\vec{F} \cdot \Delta\vec{r} = F\Delta r = Ma\Delta r$
Then we have

$$\frac{1}{2}Mv_f^2 - \frac{1}{2}mv_i^2 = Ma\Delta r$$

Or

$$\boxed{v_f^2 = v_i^2 + 2a\Delta r}$$

By not memorizing it, we make sure that we understand why the equation works and ensure that we haven't violated the 1-D assumption. Note that in 2-D $\vec{F} \cdot \Delta\vec{r}$ can equal 0 (the case of circular motion). In this case the work done is 0, and the timeless equation gives the wrong answer because we are using it inappropriately.

In the case where we have potential energy defined, there is no work done, so the equations are simpler

$$Mgh_i + \frac{1}{2}Mv_i^2 = Mgh_f + \frac{1}{2}Mv_f^2$$

from which we immediately get

$$v_f^2 = v_i^2 + 2g\Delta h$$

which is again the timeless equation, but this time in the vertical direction.

3.2 Momentum

3.2.1 Definitions

Momentum

We already have, from previous physics courses, a concept of how hard it is to accelerate an object. That concept is the mass of the object and the equation $\vec{a} = \frac{\vec{F}}{m}$.

Momentum is intended to apply this concept to stopping a moving object. First let's solve the problem intuitively.
If I am trying to stop something moving slowly, it will be easier than something moving quickly. It follows that momentum should depend on velocity.
If I am trying to stop something that has a smaller mass, it will be easier that stopping something with a larger mass. It follows that momentum should depend on mass.
From here we can use dimensional analysis to get

$$\boxed{\vec{p} = m\vec{v}}$$

Note that the equation is the definition. All the words are just to help you get an idea of what it means.

Impulse

Conservation of momentum in open systems can be derived Newton's laws and kinematics if you want. I think it is better to think of momentum as the fundamental quantity, but here is the derivation if you would rather think in terms of forces.

$$\vec{F} = m\vec{a}$$

Then we multiply both sides by t. This is equivalent to applying a force for some time and seeing what will happen.

$$\vec{F}t = m\vec{a}t$$

Now, using $\Delta\vec{v} = \vec{a}t$ from kinematics we can replace $\vec{a}t$ on the right side of the equation.

$$\vec{F}t = m\Delta\vec{v}$$

But $\Delta\vec{p} = m\Delta\vec{v} + \vec{v}\Delta m$ by the definition of momentum and the chain rule from calculus. In our case $\Delta m = 0$ So:

$$\vec{F}t = \Delta\vec{p}$$

Now we use the definition of impulse $\boxed{\vec{j} = \vec{F}t}$

And we have

$$\boxed{\vec{j} = \Delta\vec{p}}$$

This result shouldn't surprise you. Intuitively if we want to stop something large that is moving fast, we either need a very large force (like running it into a wall) or it will take a long time (like waiting for it to stop from friction).

3.2.2 Momentum and Kinetic Energy

Kinetic energy and momentum are both useful quantities for determining how an object will stop, but they have slightly different uses. The biggest difference is that momentum is a vector. This means that momentum gives us information about the direction of motion, but kinetic energy doesn't.

A result of this is that momentum can be negative. Negative momentum just means that the object is moving in the other direction. Kinetic energy can never be negative.

Mathematical Relation

Starting with

$$KE = \frac{1}{2}mv^2$$

and

$$\vec{p} = m\vec{v}$$

we can solve the momentum equation for velocity to get

$$\vec{v} = \frac{\vec{p}}{m}$$

Once we have this, we can plug it into the KE equation to get

$$KE = \frac{1}{2}m\left(\frac{\vec{p}}{m}\right)^2$$

Remember that when we square a vector we lose all information about direction. (If you forgot think about squaring -1. $(-1)^2 = 1 = 1^2$, thus we lost our direction information here also.)

Simplifying our result gives:

$$KE = \frac{p^2}{2m}$$

This means if we have mass and momentum, we can get the kinetic energy. This doesn't work the other way because KE is a scalar and doesn't have the direction information we need for momentum.

Energy gives Stopping Distance

Assume that we have an object with kinetic energy KE and a constant force F is being applied to the object in order to stop it. We have F and energy. We can relate them by using the work equation (since we only have kinetic energy, $E = KE$).

$$FD = W = \Delta KE$$

Simplifying and using that $\Delta KE = KE_i - KE_f$ where KE_f is 0 because we stop, gives

$$FD = KE_i$$

Then the stopping distance is

$$D = \frac{KE_i}{F}$$

Note that, with a constant force, **stopping distance depends on energy, but not momentum!**

Momentum gives Stopping Time

Going back to the section on impulse (3.2.1), we use the equation

$$\vec{F}t = \Delta\vec{p}$$

Since we stop at the end $\Delta\vec{p} = \vec{p}_f - \vec{p}_i = -\vec{p}_i$.
so

$$t = \frac{-\vec{p}_i}{\vec{F}}$$

If you are worried about time being negative, just remember that $\vec{p}_i$ and $\vec{F}$ are in opposite directions, so we get an extra negative sign from that.

Critically, with a constant force, **stopping time depends on momentum, but not energy!**

3.2.3 Momentum Conservation

We haven't talked yet about why momentum is so important in physics. Momentum is important because it is conserved.

- **Momentum Conservation: The total amount of momentum in the universe remains constant**
- **Mathematically:** $\boxed{\Delta\vec{p}_{tot} = 0}$

If it ever seems like the total momentum changed, that really means that it went somewhere else (like into the Earth).

3.2.4 Collisions

The classic problem using momentum conservation is 2 objects colliding. The general strategy here is to remember that total momentum cannot change. Then we find the initial total momentum, and recognize that the final total momentum must be the same.

Types of Collisions

We classify collisions into 3 types.

1. (Perfectly) Elastic Collision: A collision where kinetic energy is conserved

2. (Perfectly) Inelastic Collision: A collision where the objects stick together losing as much kinetic energy as possible.

3. Partially Elastic/Inelastic: A collision where some kinetic energy is lost, but the objects don't stick together.

There is some disagreement about what to call each of these types. I prefer to always include the "Perfectly" for clarity, but some people will just call them "Elastic" and "Inelastic." Scientists are also sloppy about whether to call the third case "Partially Elastic" or "Partially Inelastic," strictly speaking, both are correct.

If you are ever unsure about which type of collision it is, do this:

1. Check if the objects stuck together at the end (if both objects have the same final velocity, they stuck together) If so, it is perfectly inelastic.

2. Add the initial kinetic energies together, and compare to the final kinetic energy. If they are the same, it was perfectly elastic.

3. If neither of the conditions above was true, it is partially elastic.

If we know that a collision is perfectly inelastic, we can use that information to help us solve the problem. In this case we know the final velocities will be the same.

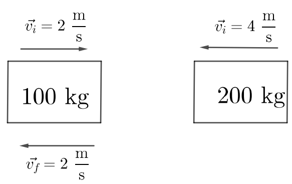
A Practice Problem

The driver of a motorcycle decides to check his text messages while driving. As a result, two motorcycles collide. The initial velocity of the first motorcycle was 2 $\frac{\text{m}}{\text{s}}$ *right and it's mass is* 100 kg*. The initial velocity of the second motorcycle is* 4 $\frac{\text{m}}{\text{s}}$ *left, and it's mass is* 200 kg*. If the final velocity of motorcycle 1 is* 2 $\frac{\text{m}}{\text{s}}$ *left, what is the final velocity of motorcycle 2?*

That's a lot of information. Let's sort it out. When we do that, we pick a direction to be positive. I always choose right as positive.

- Motorcycle 1: $M_1 = 100$ kg and $v_{1,i} = +2\ \frac{\text{m}}{\text{s}}$ also $v_{1,f} = -2\ \frac{\text{m}}{\text{s}}$
- Motorcycle 2: $M_2 = 200$ kg and $v_{2,i} = -4\ \frac{\text{m}}{\text{s}}$ also $v_{2,f} = ?$

first make a diagram.



Start by writing out that momentum is conserved

$$\vec{p}_i = \vec{p}_f$$

$$m_1\vec{v}_{1i} + m_2\vec{v}_{2i} = m_1\vec{v}_{1f} + m_2\vec{v}_{2f}$$

Now we convert to the scalar form, remembering that things moving in the + direction are positive and and things moving in the − direction are negative. This gives

$$m_1 v_{1i} - m_2 v_{2i} = -m_1 v_{1f} + m_2 v_{2f}$$

Solving this gives

$$v_{2f} = \frac{m_1 v_{1i} - m_2 v_{2i} + m_1 v_{1f}}{m_2}$$

Plugging in numbers gives

$$v_{2f} = \frac{200\ \text{kg}\frac{\text{m}}{\text{s}} - 800\ \text{kg}\frac{\text{m}}{\text{s}} + 200\ \text{kg}\frac{\text{m}}{\text{s}}}{200\ \text{kg}}$$

or

$$\boxed{\vec{v}_{2f} = -2\ \frac{\text{m}}{\text{s}}}$$

Note that in 1-D, we can use a + or - sign instead of using $\hat{x}$. In multiple dimensions, we still need to use $\hat{x}$!

3.2.5 Momentum conservation in Multiple Dimensions

The 1D equations for momentum conservation are useful, but they aren't fully general, because we live in a universe with (at least) 3 dimensions. In 3D, the momentum conservation equation takes the same vector form

$$\Delta\vec{p} = \vec{0}$$

The distinction between 0 and $\vec{0}$ is not important for physics, but in linear algebra the distinction between scalar 0 and vector 0 becomes important. The important difference in 3D is that the vector equation now is equivalent to 3 scalar equations

$$\begin{cases} \Delta p_x = 0 \\ \Delta p_y = 0 \\ \Delta p_z = 0 \end{cases} \tag{3.6}$$

All three equations must be satisfied. In an open system, it is possible for external forces to act in a way that appears to turn $\hat{\mathrm{x}}$ momentum into $\hat{\mathrm{y}}$ or $\hat{\mathrm{z}}$. An example of this is an object bouncing off of a surface that is angled at 45°. The object will bounce off the surface at a 90° angle to its original path. This appears to convert $\hat{\mathrm{x}}$ momentum into $\hat{\mathrm{y}}$ momentum, but it is only possible because momentum is transferred to and from the wall (and ultimately to the Earth). In a closed system there is no way to change the direction of the systems total momentum or its magnitude.

3.2.6 2D Momentum Conservation Example

Problem

Two objects, isolated from the rest of the world, initially have momentums given by

$$\vec{p}_{1,i} = 10\frac{\mathrm{kg\cdot m}}{\mathrm{s}}\hat{\mathrm{x}} + 10\frac{\mathrm{kg\cdot m}}{\mathrm{s}}\hat{\mathrm{y}}$$

$$\vec{p}_{2,i} = -5\frac{\mathrm{kg\cdot m}}{\mathrm{s}}\hat{\mathrm{x}} + 5\frac{\mathrm{kg\cdot m}}{\mathrm{s}}\hat{\mathrm{y}}$$

After the collision, the first object is observed to have a momentum

$$\vec{p}_{1,f} = 7\frac{\mathrm{kg\cdot m}}{\mathrm{s}}\hat{\mathrm{x}} + 11\frac{\mathrm{kg\cdot m}}{\mathrm{s}}\hat{\mathrm{y}}$$

What is the final momentum of the second object?

Solution

We first find the momentum of the whole system initially by adding up the individual momentums

$$\vec{p}_{tot} = 5\frac{\mathrm{kg\cdot m}}{\mathrm{s}}\hat{\mathrm{x}} + 15\frac{\mathrm{kg\cdot m}}{\mathrm{s}}\hat{\mathrm{y}}$$

Now the objects interact with each other (the details of the interaction aren't relevant) After the collision the system's total momentum must be unchanged. So the final momentum of the second object must be

$$p_{2,f} = \vec{p}_{tot} - \vec{p}_{1,f}$$

plugging in our values

$$\boxed{p_{2,f} = -2\frac{\mathrm{kg\cdot m}}{\mathrm{s}}\hat{\mathrm{x}} + 4\frac{\mathrm{kg\cdot m}}{\mathrm{s}}\hat{\mathrm{y}}}$$

If we were given masses of the objects, we could then calculate the final velocities.

3.2.7 Example: Conceptual Application of Conservation

The conservation laws we have learned (along with others) can be thought of as the rules for what is allowed to happen in the universe. This interpretation can be used to quickly explain why an event cannot happen. Note that saying something is not disallowed by a conservation law is not the same as saying it will happen, the universe has other rules that may prevent some events from happening.

1. *Imagine that two vehicles with masses M and $2M$ are traveling in opposite directions with the same speed, down a very icy road so that friction cannot act. The two cars collide, can they come to a stop after the collision?*

   To determine whether something can happen we use conservation laws. For collisions, energy is conserved but can change into other forms. This means that energy conservation is not particularly useful in determining whether this is possible. Let's try momentum instead.

   The initial momentum of the system is

   $$\vec{p}_i = Mv\hat{\mathrm{x}} - 2M\hat{\mathrm{x}} = -Mv\hat{\mathrm{x}}$$

   Note that the sign and the choice of unit vector is arbitrary here since no direction is given in the problem, so you could have just as easily said $\vec{p}_i = -Mv\hat{\mathrm{y}} + 2Mv\hat{\mathrm{y}}$.

   At the end we have that both cars stopped

   $$\vec{p}_f = 0$$

   Since the system was free of external forces, the initial and final momentum must be the same. In our case, they are not, so the collision is disallowed by momentum conservation.

2. *An object falls from the top of a building. Initially it is at rest, but after some time, it has a velocity of $-v\hat{\mathrm{z}}$. Is this event disallowed by conservation laws?*

   In this case, the system is open since the Earth acts as an external force. This means neither momentum nor energy are required to be conserved. The event is not disallowed based on the information presented (it might still be impossible depending on the value of v and the height of the building. Remember that conservation laws may prove that something is impossible, but satisfying the conservation laws does not prove that the event is possible).

3. *Two students with the same mass and same speed in opposite directions slide into the ends of a massless spring that is initially at rest between them. They then move in opposite directions with different speeds.*

   This violates momentum conservation. Take the system that consists of both students and the spring. Initially all objects are at rest, so they have momentum 0. At the end, they have momentum in opposite directions, but those momentums do not add to 0. Since the system is closed, this violates momentum conservation.

3.3 Angular Momentum and Rotation Intro

Up until this point we have pretended that all objects were incapable of rotating. This is actually a very good approximation in many cases, but sometimes the fact that an object is capable of rotating turns out to be important (wheels would not be very useful if they could not spin).

3.3.1 What is rotation?

The most readily available example of rotation is a wheel or gear, where an object is physically going around in a circle repeatedly. This intuition turns out to be good, rotation is just any case where points on an object travel in a circle, and a single point (or a single line if in 3-D) is kept fixed. This line is called the axis of rotation. Some examples:

1. Wheels on cars rotate around the axle, but the center of the axle doesn't move as part of this.
2. A door rotates as it swings open; the hinges stay in one place.
3. The Earth rotates once each day, with a line through the poles not moving at all.

We will only talk about rotation of solid objects as part of this. Most of this also applies to planet orbits and other systems, but in that case $\vec{\omega}$ is not the same for every orbit around the same star.

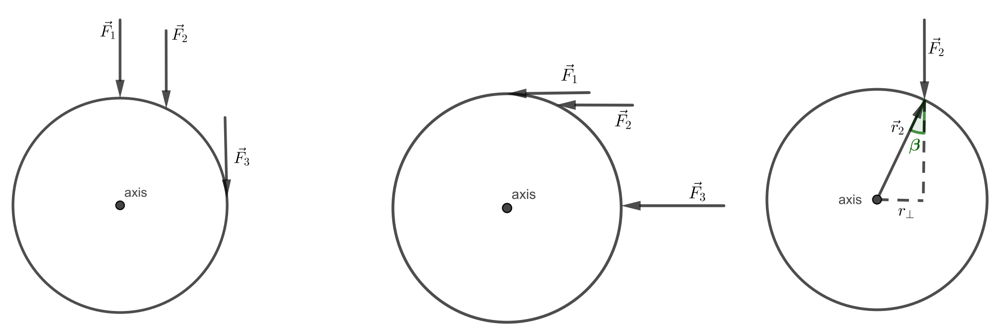
3.3.2 Force → Torque

We start with an analogue to the concept of a push or pull. In linear motion it does not matter where I push on an object, the result was always the same. Intuitively, we know that is not true if we want to make an object rotate. As an example, try to open a door by pushing right next to the hinge; you won't be able to. In fact, the further from the hinge that we push, the easier it is to open the door!

We can generalize this to any example of rotation. The further we push from the axis of rotation, the easier it is to initiate rotation.

You might have already noticed a problem with this. Imagine that I am holding an axle with a bicycle tire on it. Someone comes along and pushes straight down on the top of the tire (force $\vec{F}_1$ in figure 3.1a). This won't cause the object to rotate at all! Thinking about the other forces, those can surely cause rotation with $\vec{F}_3$ causing quite a bit more than $\vec{F}_2$. Since all the forces act a distance of the radius of the circle from the axis, it must be more than just how far we are.

Figure 3.1: A rod is placed through the center of a bicycle tire and the tire is held out. Forces are applied to the tire at different locations and angles.



(a) Forces applied from above

(b) Forces applied from left

(c) A force with its effective lever arm.

What if we changed the direction of the circle to the configuration shown in figure 3.1b? Now we have that $\vec{F}_1$ would cause the most rotation, and $\vec{F}_3$ would cause none.

From this we can start to make sense of it. Imagine a vector $\vec{r}$ that goes from the center of the circle (the axis of rotation) to the point that we are pushing on. If $\vec{r}$ is parallel (or anti-parallel) to the force, no rotation occurs. If $\vec{r}$ is completely perpendicular to the force, we get as much rotation as possible from that force. You might the the pattern here, the further that our force is from being over the axis, the more it produces rotation. Fig 3.1c shows $\vec{r}$ and the effective lever arm $r_\perp = r_2 \sin\beta$ for a single force $\vec{F}_2$. Put another way, the tendency of a force to cause rotation depends on the sine of the angle between $\vec{r}$ and $\vec{F}$. We thus define the torque as

$$\vec{\tau} = \vec{r} \times \vec{F} \tag{3.7}$$

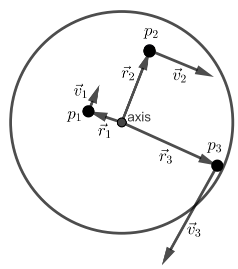
The cross product multiplies the perpendicular components of two vectors together while discarding the parallel ones. We won't be taking any cross products until we consider forces applied at angles later. For now we consider only the simple case where $\vec{r}$ and $\vec{F}$ are perfectly perpendicular, in this special case $||\vec{r} \times \vec{F}|| = rF$.

3.3.3 Angular Velocity

We define angular velocity as the rate of change of angular displacement, just like we defined linear velocity to be the rate of change of linear displacement.

What is more interesting is the relation to linear velocity. If every point on (and in) the object is going to cover the same 2π radians in the same amount of time (they must!) then they will all be covering the same number of radians each second, that is they all have the same angular velocity (denoted $\vec{\omega}$). Their linear velocities, however, are not the same, points that are further from the axis of rotation will have higher linear velocities as seen in figure 3.2.

Figure 3.2: A flat disk with 3 dots painted on it rotates on a table. The dots further away from the center have higher linear velocities than those closer to the center but the same angular velocity.



Now lets consider the Earth. A point on the surface or Earth is (roughly) the same distance from the center of Earth, but a point on the North Pole will not be moving at all as a result of the Earth's rotation. The apparent difference in behavior is because, in 2-D, the axis of rotation is just a point, but in 3-D it is a line. Once again, only the distance from the axis of rotation matters, not necessarily the radius of the object. Put another way, we take only the component of radius that is perpendicular to the axis of rotation. The component parallel to the axis contributes nothing. This explains the behavior at the north pole. At that point the vector from the center of the Earth to the object points directly along the axis of rotation. This behavior is similar to what we saw for torque, so a similar mathematical form makes sense. We can now see that the relation between linear and angular velocity also involves the cross product.

$$\vec{v} = \vec{\omega} \times \vec{r} \tag{3.8}$$

3.3.4 Moment of Inertia and Rotational Kinetic Energy

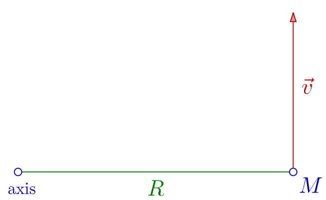
Before we can go any further, we need to define something that behaves in the rotational case like mass did in the linear case. In linear motion mass was how much an object resisted a change in its velocity. The equation for this was $\vec{F}t = m\Delta\vec{v}$ or $m = \frac{Ft}{\Delta v}$

It is logical that our rotational analogue should measure how much an object resists changes in it's angular velocity. We have chosen to call this "Moment of Inertia" and we usually denote it I in the same way that mass was denoted M.

Intuitively, we expect that I should be higher for a more massive object. It is certainly harder to make a car tire spin than a bike tire. It should also have something to do with the length of the object. It's quite easy to make things move with some specific angular velocity if they are short, much harder if they are very long.

While conceptually it was easier to think in terms or making the object start moving, getting an idea for the mathematical definition will be easier if we think using energy. If it is going to serve as a rotational analogue of mass, it should do so for all equations we know.

We will look at the energy of a mass-less rod of length R[††] with a mass M attached to the end.



The rod itself has no energy because it is mass-less. The kinetic energy of the point mass will be $\frac{1}{2}mv^2$. We recall from the previous section that $v = \omega r$. In this case the $r = R$. So we have

$$KE = \frac{1}{2}m\omega^2 R^2$$

This tells us that the energy of a particle increases quadratically as we move the mass further away from the center. If we want this I to behave in the angular case as m does in the linear case, it must also relate kinetic energy to angular speed. Thus we should have

$$KE = \frac{1}{2}I\omega^2$$

Thus we have

$$I = mR^2$$

for a single mass. What if we had multiple masses? Kinetic energy adds linearly, so the total energy is the sum of the energy of each mass. Thus the total moment of inertia is the sum of all the individual moments of inertia.

Mathematically, if there were two masses m_1 and m_2 at radius r_1 and r_2 from the center

$$KE = KE_1 + KE_2 = \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2 = \frac{1}{2}\left(m_1r_1^2 + m_2r_2^2\right)\omega^2 = \frac{1}{2}\left(I_1 + I_2\right)\omega^2 = \frac{1}{2}I_{tot}\omega^2$$

Where the last equality comes from the requirement that $\frac{1}{2}I\omega^2$ must give the kinetic energy for any rotating object.

$$I_{tot} = \sum_k I_k \tag{3.9}$$

[††]This path will trace out a circle of radius R

This works fine for point masses, but what about objects like rods, or disks? In this case we could imagine breaking the object into a huge number of parts and then adding them all together. This is possible, but obnoxious. The real answer is to use calculus, which is beyond the scope of these notes.

Instead of calculating them ourselves, I will provide a list of these here. Don't memorize them, tables exist where they can be easily looked up. All of these are taken with respect to an axis through the center of the object unless noted and constant density is assumed.

1. Solid Sphere: $\frac{2}{5}MR^2$
2. Thin Spherical Shell: $\frac{2}{3}MR^2$
3. Disk or cylinder: $\frac{1}{2}MR^2$
4. Rod (axis at one end, $\perp$ to rod): $\frac{1}{3}ML^2$
5. Rod (axis in center, $\perp$ to rod): $\frac{1}{12}ML^2$

3.3.5 Properties of Rotational Kinetic Energy

We will now formally define the rotational kinetic energy to be given by

$$KE_r = \frac{1}{2}I\omega^2$$

It is important to note here that we used the linear kinetic energy to get to this formula. This means that **there is no difference between linear and rotational kinetic energy**. They are both just kinetic energy. The use of having both is that calculating the linear kinetic energy for each point and adding them up to get the rotational energy of an object would be very annoying to do in practice.

It is also worth noting that an object that is both rotating and translating will have total kinetic energy equal to

$$KE = KE_r + KE_T = \frac{1}{2}I\omega^2 + \frac{1}{2}mv^2 \tag{3.10}$$

Where v is the speed of the center of mass, and omega is the angular speed of the object's rotation. Note that it is **NOT** necessarily true that $v = \omega r$ in this case, since the center of mass speed of an object that is both translating and rotating need not have any relation to the rotation speed. In some cases though, the two are related. That is called a no slip condition.

3.3.6 RKE Example: Objects Rolling Down A Ramp

Let's imagine that we have an object at the top of a ramp. The object has a mass M, a radius R and a moment of inertia of $I = \epsilon MR^2$ (note that it will always be some constant multiple by MR^2, so I am just taking that constant to be ϵ). We roll the object down the ramp and want to find it's final speed.

We use energy conservation

$$PE_i = KE_f$$
$$PE_i = Mgh$$
$$KE_f = \frac{1}{2}Mv^2 + \frac{1}{2}I\omega^2$$

We can simplify the rotational part of this equation.

$$KE_f = \frac{1}{2}Mv^2 + \frac{1}{2}\epsilon MR^2\omega^2$$

But $R^2\omega^2 = v^2$ so

$$KE_f = \frac{1}{2}Mv^2 + \frac{1}{2}\epsilon Mv^2$$

or

$$KE_f = \frac{1}{2}Mv^2(1+\epsilon)$$

Now we set the final kinetic energy equal to the initial potential energy

$$Mgh = \frac{1}{2}Mv^2(1+\epsilon)$$

M will cancel out, and we can solve for v to get

$$\boxed{v = \sqrt{\frac{2gh}{1+\epsilon}}}$$

We check our answer by noticing that when $\epsilon \to 0$ this reduces to $\sqrt{2gh}$, which was the answer without rotation. Notice 3 things about this answer

- M did not show up.
- R did not show up
- ϵ did show up

ϵ depends only on the shape and distribution of the mass. This means that all objects of the same shape will reach the ground at the same time, even if they are dramatically different masses or sizes. The only way we can make them reach the ground at different times is if we change the shape or mass distribution of the object. Mass distribution here refers to the way that mass is arranged in the object. As an example, we could image two cylinders, that are the same size and same mass, but one has the edge made of steel and the rest plastic, the other has the center made of lead and the rest plastic. This would result in the object with the steel edge having a substantially higher moment of inertia because a higher fraction of the mass is further away from the axis.

3.3.7 RKE Example: Object With a Force Applied to the Edge

Now let's say that we have a cylinder with a mass M, a radius R, and a moment of inertia $I = \epsilon MR^2$. We apply a constant force to the cylinder's edge (at radius R). The force is applied in a way that is always exactly tangent to the edge of the cylinder. Let's also say that the cylinder experiences no other torques or forces.

1. How fast will the cylinder be rotating when it has made 1 complete revolution?

   We use work and energy.

   $$\tau\theta = \Delta E$$
   $$\tau\theta = \frac{1}{2}I\omega^2$$

   We will make 1 revolution, so that is 2π radians. We will also use the definition of torque and the moment of inertia in this problem

   $$2\pi FR = \frac{1}{2}\epsilon MR^2\omega^2$$

   We cancel out an R and solve

   $$\boxed{\omega = \sqrt{\frac{4\pi F}{\epsilon MR}}}$$

Note that in this case, M, R, and ϵ all matter. The angular velocity will depend on all of these things! Specifically, higher R, higher M, or higher ϵ all result in slower (angular) motion. Note that if we wanted the linear velocity of a point on the edge, we would use $v = \omega R$ to get

$$v = \sqrt{\frac{4\pi FR}{\epsilon M}}$$

So a point on the edge is actually going faster for the object with a larger R, even though the angular velocity is less

2. How fast will the cylinder be rotating at time t?

   We use angular momentum and angular impulse

$$\tau t = \Delta L$$

$$FRt = I\omega$$

$$FRt = \epsilon MR^2\omega$$

$$Ft = \epsilon MR\omega$$

$$\boxed{\omega = \frac{Ft}{\epsilon MR}}$$

   Once again, M, R, and ϵ all matter. Specifically, higher R, higher M, or higher ϵ all result in slower (angular) motion. If we look for the linear velocity again, we get

$$v = \frac{Ft}{\epsilon M}$$

   So radius is irrelevant if we are looking for the velocity at some time here.

3.3.8 Angular Momentum

If everything else has an analogue, we expect momentum to as well. The analogue of momentum is angular momentum, usually denoted $\vec{L}$. If we want a quantity that functions the same way as momentum, we need something that follows the same rules as linear momentum, that is

$$\Delta \vec{L}_{tot} = 0 \tag{3.11}$$

for any closed system, and

$$\Delta \vec{L} = \vec{\tau} t \tag{3.12}$$

for a system that is subjected to external forces. Fortunately, we can quickly find something that behaves like that.

$$\vec{L} = I\vec{\omega} \;\; ^{\ddagger\ddagger} \tag{3.13}$$

Once again, a perfect analogy to the linear case.
You might also notice that, since $I = MR^2$, we can write $I\omega = MvR$, but Mv is linear momentum, so

$$\vec{L} = \vec{r} \times \vec{p} \tag{3.14}$$

The simplicity of angular momentum should not be taken to mean that it is not useful. The importance of conservation laws in physics cannot be overstated and, like linear momentum, it often provides an easy way of understanding or solving a problem.

[‡‡]Quick derivation $\vec{\tau} = I\vec{\alpha}$ multiplying by t gives $\vec{\tau} t = I\vec{\omega}$

Chapter 4

Forces and Newton's Laws

4.1 The fundamental forces

Here I provide the fundamental forces in order of strength. The weak force is actually stronger than the EM force under certain very specific conditions, but otherwise the relative strength of the forces hold. We won't deal with the cases where the weak force is stronger here. Note that these are the only forces that really exist. The other forces are actually the result of the fundamental forces.

4.1.1 The Strong Force

1. The strong force is the force that binds quarks together. The left over strong force creates the nuclear force, which binds atomic nuclei together. The strong force has these properties when acting between free quarks:

   (a) Infinite range

   (b) Force does **not** decrease with distance (after a certain threshold).

   (c) Relative strength: 1 (by definition)

   The strong force is the strongest force

   (d) Acts to bind quarks together

   Nuclear Strong Force

   The nuclear strong force has these properties:

   (a) Short range

   (b) Force decreases rapidly with distance

   (c) Can still only act on particles made of quarks (usually protons and neutrons).

   (d) Is responsible for holding atomic nuclei together.

   (e) Strongly repulsive at distances less than about 0.75×10^{-15}m

      i. Repulsion becomes stronger as distance decreases

   (f) Attracts more strongly than the EM repulsion for distances between about 0.75×10^{-15}m to about 2.5×10^{-15}m

      i. This property allows nuclei to exist.

      ii. The extremely rapid drop off limits the size of atomic nuclei.

4.1.2 The Electric force

The electrical force (aka Coulomb Force) is the force that exists between charged objects. It has the following properties:

1. Infinite range
2. Force decreases with distance.
3. Relative strength: $\sim 10^{-2}$ (compared to strong force)
4. Results from interactions between photons and charged particles
5. Equation:
$$F_c = \frac{kq_1q_2}{r^2} \tag{4.1}$$
   (a) Like charges repel
   (b) $k = 8.99 \times 10^9$ N m^2 C^{-2}

4.1.3 The Weak Force

The weak force governs interactions between certain subatomic particles.

1. Very short range.
2. Force decreases extremely rapidly with distance.
3. Relative strength (for most purposes): $\sim 10^{-5}$ (compared to strong force)
4. Primarily important for subatomic decays.
5. Typically only relevant when the strong and EM forces are prevented from acting by some conservation law.

4.1.4 The Gravity "Force"

The "force" that exists between any two objects with mass

1. Infinite range
2. Force decreases with distance
3. Relative strength $\sim 10^{-39}$ (compared to strong force)
4. Least understood force.
5. May or may not actually be a force.
6. Appears to be only attractive (we haven't observed otherwise)
7. Equation (for the purpose of this document):
$$F_g = \frac{GM_1M_2}{r^2} \tag{4.2}$$
   (a) We will often use the acceleration due to gravity $g = 9.81\frac{\text{m}}{\text{s}^2}$ (for Earth)
   (b) $G = 6.67 \times 10^{-11}\ \frac{\text{m}^3}{\text{s}^2 \cdot \text{kg}}$

Solid Objects

Think for a moment about why objects seem solid. They are composed of atoms that are nearly all empty space. When you push on one, what is really happening is that the electric force is repelling your hand so strongly that the object feels solid.

4.2 Forces and Newton's Laws

4.3 Basic Contact Forces

1. **Applied Force**: The force on an object applied by a person or another object.
   (a) Often denoted by F_a
   (b) Can push or pull.
2. **Normal Force**: The perpendicular force that is applied by any surface that an object is pushed or pulled against.
   (a) Denoted by F_N
   (b) Always pushes, cannot pull.
   (c) Direction is always exactly perpendicular to the contact surface. Note, this is not always straight up.
   (d) Magnitude must be calculated from the other forces on the object.
3. **Tension Force**: The force that results from a tight rope or string pulling on an object
   (a) Denoted by F_T or T
   (b) Always pulls, cannot push.
   (c) Magnitude is equal to the magnitude of the force on the other end of the rope.

4.4 Gravity

4.4.1 Basics of Gravity

Essentially, gravity can be summarized by saying that all massive objects attract each other. We certainly don't notice the gravity of a car (because it is very small) but for very large objects or over very large scales, gravity dominates.

1. Weakest fundamental force by a huge factor ($\sim 10^{39}$ times weaker than the strong force)
2. Not well understood at the microscopic level, but easy to observe
3. Because it can only be attractive, it is the only force that does not cancel on large scales
4. Force law:
$$F_g = \frac{GMm}{R^2} \tag{4.3}$$
   - $G = 6.67 \times 10^{-11}\ \frac{\mathrm{m}^3}{\mathrm{kg \cdot s}^2}$
   - M and m are the masses in kg
   - R is the distance from the center of one object to the center of the other in meters

If it is hard to see gravity should win on large scales, consider why the other forces can't win. The weak force and residual strong force are obvious: they rapidly fall off with distance. The true strong force seems like it should be important because it doesn't fall off with distance at all, but it is so strong that when pulled apart, the energy quickly becomes large enough to form new particles, ending any ability to act on long distances. That leaves electromagnetism. Electromagnetism can be both attractive and repulsive. It is strong enough that when positive or negative charges are isolated, they will rapidly recombine even over large distances. This means that matter is neutral and thus experiences very little action from the electric force. If gravity were somewhat stronger, it would have attracted all the matter in the universe together.

4.4.2 What is Gravity?

Unlike the other fundamental forces, gravity does not appear to act like a force at all on large scales. Instead it behaves like a change in the definition of what a straight line is. Imagine that you are tasked with drawing a straight line on the surface of a basketball. You could do your best by drawing a circle whose center is the center of the basketball. Now you draw another straight line (also a circle whose center is the basketball's center). Note that these two lines, which seemed parallel at the equator, converge at the poles. There are no true perpendicular or "straight" lines on the surface of a sphere. Some appear straight and parallel when we zoom in far enough, but if we look at the whole object, they aren't.

It seems that massive objects alter the definition of a straight line in a similar way. "Straight" lines stop existing near massive objects. The lines appear straight when we are near the object, but when we zoom out, we see that they aren't. Don't worry too much if this is complicated. For many applications *, treating gravity as a force entails only a negligible error. Since these notes are not meant to cover general relativity, we will treat gravity as a force here.

4.4.3 The two force laws for gravity

You have probably seen in your textbook or lecture that the gravitational force on an object is given by

$$F_g = Mg \tag{4.4}$$

Strictly speaking, this isn't true. Equation (4.3) is the general force law for gravity, (4.4) is only valid when we are on the surface of Earth. Let's investigate why.

We start with

$$F_g = \frac{GMm}{R^2}$$

Let's take the values for Earth. Then this is

$$F_g = \frac{GM_e m}{R_e^2} \tag{4.5}$$

We assumed here that $R = R_e$. This is true only when we are on the surface of Earth.

We then note that G, M_e and $\mathrm{R_e}$ are all constants.

Constants are just numbers with units. If we have any constants we can be sure that if we multiply and/or divide them, we will get something that is still a number with units.

This means that $\frac{GM_e}{R_e^2}$ is some constant. Let's plug in the numbers and see what that constant is.

We look up the values of each constant and find that:

*in addition to extreme cases like neutron stars or cosmology, GPS actually requires a geometric treatment due to high accuracy requirements

- $G = 6.67 \times 10^{-11} \ \frac{\text{m}^3}{\text{s}^2\text{kg}}$
- $M_e = 5.97 \times 10^{24}$ kg
- $R_e = 6371 \text{ km} = 6.371 \times 10^6$ m

$$\frac{GM_e}{R_e^2} = \frac{(6.67 \times 10^{-11} \ \frac{\text{m}^3}{\text{s}^2\text{kg}}) \cdot (5.97 \times 10^{24} \text{ kg})}{(6.371 \times 10^6 \text{ m})^2}$$

It is often helpful when given a nasty formula like this to separate the equation into a product of nicer equations. We never need to do this, but it will make it easier to see what is going on. In this case we can make:

$$\frac{GM_e}{R_e^2} = \frac{6.67 \cdot 5.97}{6.371^2} \times \frac{10^{-11} \cdot 10^{24}}{(10^6)^2} \times \frac{\frac{\text{m}^3}{\text{s}^2\text{kg}} \cdot \text{kg}}{\text{m}^2}$$

Now we multiply to get

$$\frac{GM_e}{R_e^2} = \frac{39.82}{40.59} \times \frac{10^{13}}{10^{12}} \times \frac{\text{m}}{\text{s}^2}$$
$$\frac{GM_e}{R_e^2} = .981 \times 10 \times \frac{\text{m}}{\text{s}^2}$$
$$\frac{GM_e}{R_e^2} = 9.81 \ \frac{\text{m}}{\text{s}^2}$$

You should recognize this as the gravitational acceleration on the surface of Earth. We call this g.

$$\boxed{g := \frac{GM_e}{R_e^2} = 9.81 \ \frac{\text{m}}{\text{s}^2}}$$

In the equation above, := just means that we define g in this way

This means that we can write equation (4.5) as

$$F_g = m\frac{GM_e}{R_e^2} = mg \tag{4.6}$$

4.4.4 When can we use the simpler force law?

When we were getting to equation (4.6) we used that $R = R_e$. This is only exactly correct when we are exactly on the surface of Earth. Intuitively it should be good enough when we are very close to the surface of Earth. What if we want to know how far off we would be by using $F_g = mg$?

We can figure that out with division. The approximate formula is given by

$$F_{ap} = mg = m\frac{GM_e}{R_e^2}$$

The exact formula is

$$F_{ex} = m\frac{GM_e}{R^2}$$

If we want to know how far off we are, we can divide the two. If the answer is close to 1, we are fine to use the approximate formula $f_{ap} = mg$. Otherwise we need to use $F_{ex} = m\frac{GM_e}{R^2}$.

$$\frac{F_{ap}}{F_{ex}} = \frac{\cancel{m}\frac{\cancel{GM_e}}{R_e^2}}{\cancel{m}\frac{\cancel{GM_e}}{R^2}} = \frac{R^2}{R_e^2} = \left(\frac{R}{R_e}\right)^2$$

Which gives us

$$\boxed{\frac{F_{ap}}{F_{ex}} = \left(\frac{R}{R_e}\right)^2} \tag{4.7}$$

Equation (4.7) provides a convenient test for whether or not we can use the simpler formula. If $\left(\frac{R}{R_e}\right)^2$ is close to 1 then our use of $F = mg$ will not involve much error, but if $\left(\frac{R}{R_e}\right)^2$ is not close to 1, then we cannot use $F = mg$.

If R is close to R_e then R can be stated as

$$R = R_e + R_1$$

When R_1 is very small we can use that

$$\left(\frac{R}{R_e}\right)^2 = \left(\frac{R_e + R_1}{R_e}\right)^2 = \left(1 + \frac{R_1}{R_e}\right)^2$$

since $\frac{R_1}{R_e} << 1$ we can approximate this as

$$\frac{F_{ap}}{F_{ex}} \approx 1 + 2\frac{R_1}{R_e}$$

4.5 Newton's Laws

4.5.1 Newton's First Law ("Law of inertia")

"An object at rest stays at rest and an object in motion stays in motion with the same speed and in the same direction unless acted upon by an unbalanced force"

There are many equivalent ways to write this. Two examples are: **"Velocities don't change without net forces"** or (my favorite) **"Only unbalanced forces cause accelerations."**

4.5.2 Newton's Second Law

$$\vec{F}_{net} = m_{tot}\vec{a} \tag{4.8}$$

This tells us that forces will cause objects to accelerate, and that more massive objects accelerate less given the same force.

4.5.3 Newton's Third Law

Every force has an equal and opposite force.

This law tells us that if object A creates a force on object B, object B creates an equal force in the opposite direction back on object A.

4.5.4 A Note on the First and Second Law

The first law seems unnecessary in light of the second since if we take $\vec{F}_{net} = 0$ we would get $\vec{a} = 0$ as the only solution to the second law. In this context, the utility of the form **"Only unbalanced forces cause accelerations."** becomes obvious. The second law is saying that a force leads to an acceleration. The first is saying that nothing else behaves like a force as long as we are in an inertial frame.

4.6 Friction

Friction refers to forces that result from rough surfaces that are in contact with each other.

4.6.1 Friction Coefficients

Friction coefficients are measures of the roughness and chemical makeup of a surface.

1. The kinetic friction coefficient (μ_k) is used when an object is sliding across a surface.
2. The static friction coefficient ($\mu_{s,max}$) is used when an object is at rest on a surface
3. The static friction coefficient is usually larger than the kinetic friction coefficient ($\mu_{s,max} > \mu_k$)
4. For most materials the friction coefficients are both between 0 and 1 ($0 \leq \mu \leq 1$)

4.6.2 Kinetic Friction

Kinetic Friction is the frictional force that results when 1 object is sliding across another object. It has the following properties:

1. It's direction always opposes the velocity of the object
2. Results from small imperfections in the surface hooking on each other.
3. Magnitude given by
$$F_{f,k} = F_N \mu_k \tag{4.9}$$
μ_k is the coefficient of kinetic friction.
4. Can make objects accelerate, but only to slow them down relative to the surface.

4.6.3 Static Friction

1. Exists when there is an outside force attempting to make an object move and the object is still at rest.
2. Stronger than kinetic friction.
3. Direction always opposes whatever force is attempting to cause the object to slide/roll.
4. **Maximum** magnitude given by
$$F_{f,s,max} = F_N \mu_{s,max} \tag{4.10}$$
5. Actual magnitude is equal to the magnitude of the opposing force.
6. Cannot cause objects to accelerate!
7. If the maximum magnitude of the static friction is less than the force that is trying to accelerate the object, the object will start to accelerate, and we will switch to kinetic friction.

4.7 Free Body Diagrams

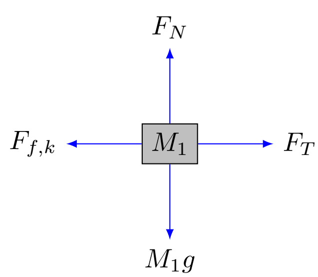
4.7.1 Rules for free body diagrams

You must follow all these rules when drawing a free body diagram. If you don't follow them, what you drew is not a free body diagram at all.

1. Start by drawing either a dot or a box to represent the mass
2. All arrows must begin on the mass (or dot) and point away from the mass.
3. Only the object you are drawing the free-body diagram for should be included
   (a) Other masses should not be included
   (b) Walls, floors etc. should not be included
   (c) Ropes should not be included
4. Only forces should appear as arrows.
   (a) Accelerations do not belong in free body diagrams (except when multiplied by masses like M_1g)
   (b) Velocities do not belong in free body diagrams.
5. Include all forces on the object in the free body diagram.
6. You may use different length arrows for different strength forces, but it isn't usually required
7. There are multiple conventions for free body diagrams. You can use any as long as you are consistent.

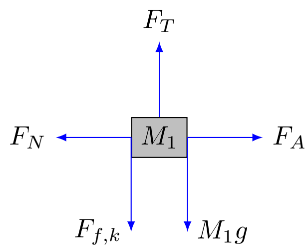
4.7.2 Drawing a Free-Body Diagram

You are pulling horizontally on a sled with mass M_1. As a result of your pulling, the sled is accelerating slowly. Draw a free-body diagram of the situation.



4.7.3 Interpreting a Free-Body Diagram

The free body diagram for a mass that is being held against a wall and hoisted upwards at constant speed is given. Use the diagram to derive an equation for the tension force on the object in terms of only M_1,g, and μ_k.



We have not net force in the x direction (because we are told that the mass is being held against the wall and is thus not accelerating). This means that our left and right forces are equal. Our equations is

$$F_N = F_A$$

We can use that to solve for the friction force.

$$F_{f,k} = F_N \mu_k = F_A \mu_k$$

Since we are told that the mass is moving with constant speed in the vertical direction we also have no net force in the y direction. This means our upward and downward forces are equal. Then our y equation is

$$F_T = F_f, k + M_1 g$$

simplifying gives

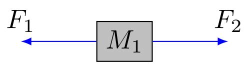
$$\boxed{F_T = F_A \mu_k + M_1 g}$$

4.8 Analyzing Complex Systems

4.8.1 The Same Thing Many Times

The Most Simple Case

We have a mass M_1 and we apply two forces to it as indicated in the free body diagram. Find the acceleration of the mass.



We find the acceleration using $\vec{F}_{net} = m_{tot}\vec{a}$ where $\vec{F}_{net}$ is the vector sum of the forces on the object. In our case there are only the two forces. So this means that

$$\vec{F}_{net} = \vec{F}_1 + \vec{F}_2 \tag{4.11}$$

To turn the vector equation into a scalar equation, we just use the magnitude of the forces. Since $\vec{F}_1$ and F_2 point in opposite directions, they will oppose each other. To show this we write the magnitudes subtracting rather than adding.

$$F_{net} = F_2 - F_1$$

Applying $a = \frac{F_{net}}{M_{tot}}$ gives

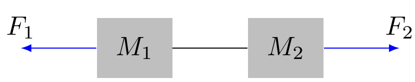
$$a = \frac{F_2 - F_1}{M_{tot}} \tag{4.12}$$

In our case $M_{tot} = M_1$, so this is just

$$\boxed{a = \frac{F_2 - F_1}{M_1}} \tag{4.13}$$

Seeing Double

Now imagine that there is a (massless) string tying two masses together. We pull on the masses with forces as indicated in the diagram below. Find the acceleration of the system.



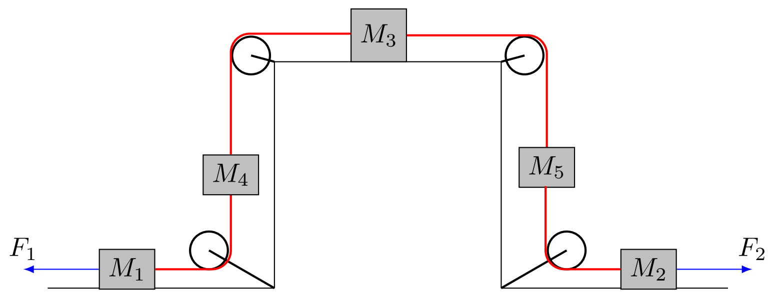
We don't need to go back to eq (4.11). We can start directly from eq (4.12) because the forces are the same.

In this case $M_{tot} = M_1 + M_2$. We plug this into equation (4.12) to get

$$\boxed{a = \frac{F_2 - F_1}{M_1 + M_2}} \tag{4.14}$$

This equation works as long as the forces oppose each other. If the forces are in the same direction there could be slack in the rope, which would make the masses accelerate at different rates.

Seeing Many Things



A Simplified Case

Well... that is a lot of masses... Let's start with a simpler case where $M_4 = M_5 = 0$. Then the problem is the same as before and eq (4.12) applies with $M_{tot} = M_1 + M_2 + M_3$. So the acceleration is

$$\boxed{a = \frac{F_2 - F_1}{M_1 + M_2 + M_3}} \tag{4.15}$$

No Slack For Students

Now we drop any assumptions about the masses of objects.

In this case we know that M_4 and M_5 will produce downward forces on the rope. It is also possible with certain forces and masses that there could be slack in the ropes caused by this. Let's assume that there isn't slack in the ropes for now.

The downward force due to gravity on the masses are $M_1 g$ and $M_2 g$ respectively. Let's define g to be positive. There are 2 forces that would tend to make the system accelerate to the right. They are F_2 and $M_5 g$. There are also two force that would make us accelerate to the left. They are F_1 and $M_4 g$. Thus

$$F_{net} = F_2 + M_5 g - F_1 - M_4 g = F2 - F_1 + (M_5 - M_4)g$$

$$M_{tot} = M_1 + M_2 + M_3 + M_4 + M_5$$

Applying $F_{net} = m_{tot} a$ gives:

$$\boxed{a = \frac{F2 - F_1 + (M_5 - M_4)g}{M_1 + M_2 + M_3 + M_4 + M_5}} \tag{4.16}$$

We want to determine if this formula makes sense. We start by examining the units. The bottom is all masses, and the top is all forces. A force divided by a mass has units of acceleration, so we are good there. Note that there are almost never any cases where we should be adding things with different units. If you see this in a formula, it is wrong.

Then we look at our equation in some simpler cases. First take the case where $M_4 = M_5 = 0$. This should reproduce equation (4.15). Inspection shows that it does.

Now take the case where $M_4 = M_5 \neq 0$. In this case we should see that the gravitational forces cancel out, but the masses remain for the purpose of the total mass. Inspections shows this to be correct as well. We thus believe that the formula we have is correct.

4.9 Non-inertial frames and pseudo-forces

Any reference frame that is accelerating will not obey the law of inertia. For this reason, we refer to such frames as non-inertial.

As a simple example, take a car with a friction-less object on the floor (if you can't imagine a frictionless object, imagine a ball rolling instead, but you'll need to ignore friction to simplify things). The object slides to the back as the car accelerates forwards and slides to the front as the car stops. We want to think about the car stopping from two reference frames.

4.9.1 Outside the car

This one is pretty easy. Someone outside sees the car accelerating in the negative direction as you stop. He would then say that the ball was actually moving with a constant velocity the entire time, the car just accelerated in the negative direction, so the ball ran into the front of the car.

4.9.2 inside the car

To you, inside the car, the objects initial velocity was 0 while it's final velocity was > 0. Thus, from your perspective, it would seem that a force must have been applied to the object to make it accelerate forward as you stopped. If you drew a free body diagram, you would account for all known forces and notice that there was no force on the object in the direction that it accelerated. You could handle this in two ways,

- You could combine this with other observations to come to the conclusion that your frame was accelerating.
- You could ignore all other data and say that another force must have been acting on the object to cause it to accelerate.

In the first case, Newton's First Law does not apply, so the behavior is not unexpected since objects are free to accelerate without forces when viewed from an accelerating frame.

In the second case, Newton's First Law would be presumed to hold, and you would simply say that there was a force without apparent origin that caused the object to accelerate. The force that you need to invent to make your reference frame behave according to Newton's Laws is called a "psuedo-force." Sometimes, very annoyingly, people will call this a fictitious force. I really don't like that term because in your frame of reference, the force you invented can absolutely cause things to happen in your reference frame. After all, you only invented it to explain something that you observed. **In an accelerating frame, psuedo-forces behave exactly like any other force, but when asked to identify the source, there is no apparent answer.**

4.9.3 Accelerations and the Equivalence Principle

The fact that pseudo-forces are real in the reference frame that is accelerating allows for a really cool trick. Basically, we can, by switching frames, turn any acceleration into a psuedo-force.

A particularly useful example is gravity and acceleration. Imagine a man in an elevator which is accelerating upwards. The equivalence principle states that this is identical to a gravity field. More generally the equivalence principle tells us that gravity is indistinguishable from acceleration.

An easy application of this is placing a system in an accelerating elevator. In that case, the object will behave like it would on Earth if we just define the effective gravity as the sum of the actual gravity and the acceleration that is in the opposite direction as the gravity.

This also has another weirder implication. If an elevator is accelerating downward at a rate that exactly matches gravity, it will cancel out gravity and produce an inertial reference frame with no apparent forces at all! To someone in free-fall, gravity is irrelevant (until they hit the ground).

If you don't believe any of this, remember that you can always do the experiment and see.

4.10 Inclined Planes

You may have noticed that the world isn't flat. We regularly need to deal with objects that are siting on inclined surfaces. At the simplest level, on an inclined plane, gravity is no longer perfectly perpendicular to the ramp. Recall from section 3.1.3 that the work done by a conservative force (such as gravity) is $\vec{F} \cdot \Delta\vec{r}$. Since the plane is inclined, the dot product of $\vec{F_g}$ and $\Delta\vec{r}$ no longer vanishes as the object moves. This means that gravity can do work on the object and accelerate it down the ramp. If we prefer to think in terms

4.11 Statics

The study of forces and torques in stationary objects is termed statics. Imagine that we want something to remain static. This is really two separate conditions. It must not accelerate linearly and it must not accelerate angularly. Consider an object with no initial motion. Conceptually it makes sense that if no outside forces whatsoever act on an object (and it doesn't explode) it must remain stationary forever. This is certainly true. Isolated objects cannot begin translating or begin rotating. When considering only linear accelerations, we could also have an object remain stationary if all forces balance.

Now consider the case where the object may rotate. It is possible for the object to rotate while having a net force of 0 because forces applied off center lead to torques. Forces balancing does not mean that torques will also balance. We will need to get a bit more mathematical to understand equilibrium conditions in this situation.

4.11.1 The Rules

Recall from previous sections that there is a relationship between force and acceleration

$$\vec{a} = \frac{\vec{F}_{net}}{m}$$

and between torque and angular acceleration

$$\vec{\alpha} = \frac{\vec{\tau}_{net}}{I}$$

We need both $\vec{\alpha} = 0$ and $\vec{a} = 0$.

Since m and I are never infinite, this condition requires that $\vec{F}_{net}$ and $\vec{\tau}_{net} = 0$ Plugging in our definitions, we have equilibrium when

$$\begin{aligned} \sum_i \vec{F_i} &= 0 \\ \sum_i \vec{\tau_i} &= 0 \end{aligned} \tag{4.17}$$

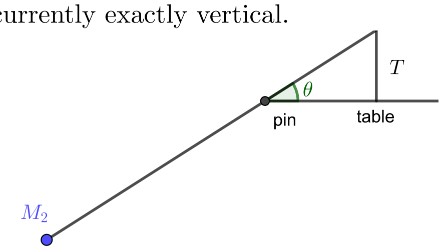
Remember that these are vector sums, so we can't forget signs! Forces that are in opposite directions cancel, torques that would cause opposite rotations cancel.

4.11.2 A Very Detailed Example

Problem statement

This section will go through an example problem and include some general tips for solving these sorts of problems.

Consider a rod with mass M_1 and length L. The rod is pinned to the edge of a table at a point $\frac{2}{3}$ of the way along its length. At one end of the rod is a mass M_2, and at the other end, a string is attached. If the object is going to remain static, what will be the tension (T) in the string? Assume that the string is currently exactly vertical.



The first step is to chose an axis of rotation. Technically we can take any axis, but most are usually not good choices. The thing to remember here is that forces that pass through the chosen axis cannot create torques around that axis. Here are the considerations (roughly) in descending order of importance. Note that these are suggestions, exceptions exist.

Choosing a pivot

1. If the problem involves a pin or hinge, calculating the normal force from the pivot will often be very difficult. Avoid this by choosing the pin.
2. A good choice of pin eliminates as many torques as possible.
3. If no obvious choice exists, choose the center of mass or an endpoint of the rod.

Following our first rule, we choose the pivot to be the axis of rotation.

Torque Balance

We now use that the net torque must be 0 for the object to remain static. Note that force balance provides us nothing useful here because the second unknown force (the normal force from the pivot) can create no torque. The three torques that result come from gravity on the center of mass of the bar (ccw), the weight of M_2 (ccw) and the tension force from the string (cw).

I will start with the torque from M_2. The equation for torque is

$$\vec{\tau} = \vec{r} \times \vec{F}$$

Where $\vec{r}$ is the vector that points from the axis to the point where the force is applied. It points along the rod here.

Recall that the pivot is at $\frac{2L}{3}$, since M_2 is at the end, this is also the distance from M_2 to the axis. Since the force $M_2 g$ points straight down, it is not perpendicular to the rod, thus the cross product will give us a trig function. To determine whether this will be sin or cos, I like to use limiting cases. Consider $\theta \to 0$. In this case $M_2 g$ and the rod are perpindicular, so the torque is large. Since cos has a maximum at $\theta = 0$, while sin has a minimum, cos is the right function. Thus

$$\tau_2 = \frac{2M_2 gL\cos\theta}{3}$$

Now we do the same thing for the torque form the string tension force, remembering that it is $\frac{L}{3}$ away from the axis to get

$$\tau_T = \frac{TL\cos\theta}{3}$$

Finally, we find the torque from gravity on the rod. The center of mass of the rod is at $\frac{L}{2}$, while the pivot is at $\frac{2L}{3}$. Thus the distance from the axis is $\frac{2L}{3} - \frac{L}{2} = \frac{L}{6}$ Using the same process as above

$$\tau_g = \frac{M_1 g L \cos\theta}{6}$$

Now we set the clockwise and counter-clockwise torques equal

$$\tau_g + \tau_2 = \tau_T$$

$$\frac{2M_2 g L \cos\theta}{3} + \frac{M_1 g L \cos\theta}{6} = \frac{TL\cos\theta}{3}$$

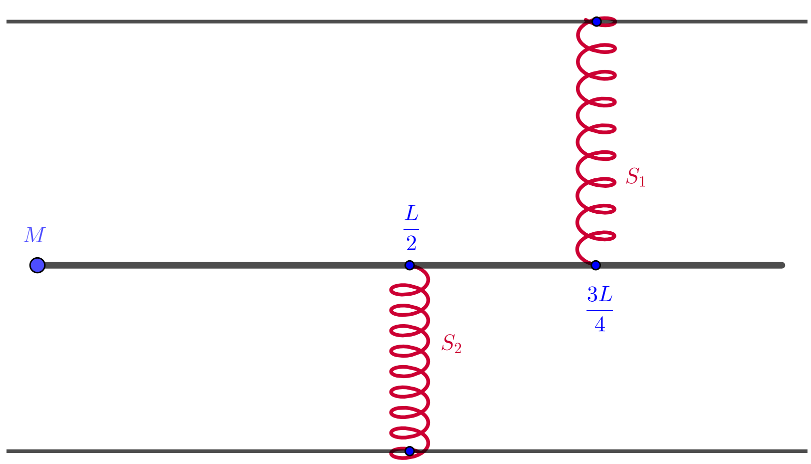
Now we solve for the tension to get

$$\boxed{T = 2M_2 g + \frac{1}{2}M_1 g}$$

4.11.3 Another Example

4.11.4 Problem Statement

Consider a bar with mass M. On one end of the bar is a mass, also with mass M. 2 springs are attached. One (S_2) from below at $\frac{L}{2}$ and one S_1 from above at $\frac{3L}{4}$. Both springs have an **unknown** spring constant k. If spring S_1 is compressed by Δx_1, find the force exerted on the ground by S_2.[†]



4.11.5 Thoughts and Setup

First notice that the downward force that S_2 applies to the floor is the same as the force it applies upward to the beam. This would be given by $F_2 = k\Delta x_2$. However, both k and Δx_2 are unknown. This means we will need either one very clever equation (where $k\Delta x_2$ is the variable we are solving for directly), or two easy ones to solve this. Note that $F_2 > 2Mg$ since S_1 is definitely applying some downward force to maintain torque balance.

I'm going to avoid the clever way specific to this problem and just use the two equation method since it works for all problems like this. Our two equations will be torque balance and force balance.

[†]I modified the springs in this diagram from work by "PaulR" available at https://help.geogebra.org/topic/a-spring-segment

4.11.6 Torque Balance

Using our rules above, the best choice of axis would be the one that eliminates as many forces as possible. Both the force from F_1 and the force of gravity act at the center of mass of the beam. Let's take that as the axis. This gives

$$\tau_m = \tau_1$$

or

$$k\Delta x_1 \frac{L}{4} = Mg\frac{L}{2}$$

Which we solve to get

$$k = \frac{2Mg}{\Delta x_1} \tag{4.18}$$

4.11.7 Force Balance

Now we use that the forces have to balance. The forces come from the mass on the end, gravity on the beam, compression of S_2 and compression of S_1. Only the force from S_2 points upward, so

$$Mg + Mg + k\Delta x_1 = k\Delta x_2$$

4.11.8 Finishing

Just algebra from here. We substitute k into the second equation to get

$$2Mg + 2Mg = 2Mg\frac{\Delta x_2}{\Delta x_1}$$

Solving gives

$$\Delta x_2 = 2\Delta x_1$$

We can use this along with equation (4.18) to get

$$\boxed{F_2 = 4Mg}$$

Chapter 5

Motion

5.1 Motion Graphs

5.1.1 The basics

When we observe an object moving, we can readily tell the speed and direction of motion, and where the object is located. These are, of course, the velocity and current position of the object. The object will also have some acceleration, but that is much harder to see visually. We want to plot all three of the quantities so that we have a full description of the motion of the object. It is also important to remember that the graphs are related to each other. This will let us use one graph to determine the behavior of the others.

5.1.2 Possible and Impossible Graphs

Generally, there are rules that all objects must follow as they move. Here are the rules and what they mean about the graphs.

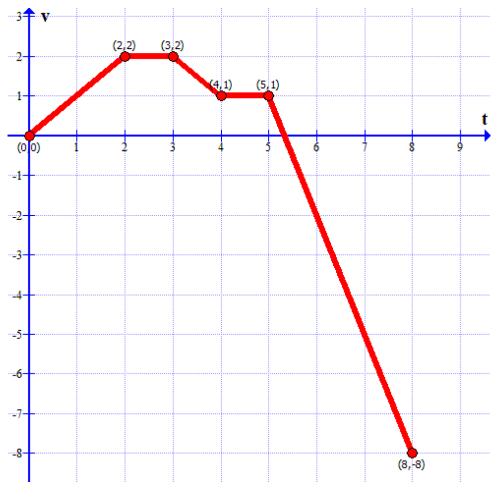
- Objects cannot teleport, so the position graph must be continuous
- Momentum must be conserved, so the velocity graph must be continuous and the position graph must be smooth

We might be tempted to write also that the acceleration must be continuous. This is true, but we will sometimes pretend that acceleration is not continuous. In the real world, acceleration cannot change *instantly*, but it can change very quickly without violating any physical laws. It is thus a good approximation in many cases to just pretend that the acceleration changed instantly. If this bothers you, remember that most of physics is really just approximations. We must choose reasonable approximations if we want our results to be reasonable, but we should not strive for perfection if it unreasonably impacts difficulty. We are aiming for the simplest approximation that is "good enough" for whatever it is we are doing.

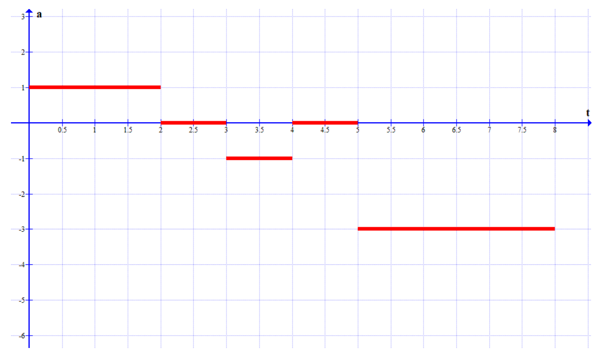
5.1.3 Basic Relations Between Graphs

For those who have had calculus. You likely know that velocity is the time derivative of position, and that acceleration is the time derivative of velocity. If you haven't had calculus, an equivalent statement is that the slope of the position graph is the velocity, and the slope of the velocity graph is the acceleration. Let's illustrate with a simple example. For clarity, the whole example will be on the next page.

We are given the following graphs, how would we use each graph to draw the other?



(a) Velocity



(b) Acceleration

Note, for clarity, I am omitting units here. Remember that velocity and acceleration both always have units! Those units might be arbitrary in some cases. To do this, we first look at one of the graphs. Remember that acceleration is the slope of velocity (alternatively, velocity is the area under the acceleration up to that time).

Acceleration to velocity

Let's pretend we have only the acceleration from time $t = 0$ to time $t = 4$. We can't consider this whole region together because the acceleration graph has breaks, so we divide it up into three intervals $0 < t < 2$ $2 < t < 3$ and $3 < t < 4$. Then we will use $\Delta v = at$ to find the changes in velocity. We can calculate the area under the graph in the first interval to be 2* This gives the change in the velocity between time $t = 0$ and time $t = 2$ to be $\Delta v = 2$. So we plot $(t = 2, v = 2)$ as a point on the velocity graph. Since the slope was constant, we plot a straight line between them.

Now look at the next section, from time $t = 2$ to time $t = 3$. During this time acceleration was 0, so $\Delta v = 0$. Since we were already at $v = 2$, we will stay there for the whole interval thus we plot a point $(3, 2)$ on the velocity graph. Since the slope is constant (specifically 0), we plot a straight line between those points.

Finally we look at the third segment, from $t = 3$ to $t = 4$. For this segment, the acceleration is $a = -1$ for the entire time. This means that we have $\Delta v = -1$ during the interval. Since we started at $v = 2$, we will end at $v = 1$. We plot that point and once more connect them with a straight line.

Velocity to Acceleration

Now let's consider the interval $4 < t < 8$ and pretend we have only the velocity graph. Once again, we divide the interval at any break points. In this case that give 2 intervals, $4 < t < 5$ and $5 < t < 8$.

*Note that if the units of acceleration were $\frac{m}{s^2}$, we would be multiplying this by t in seconds to get units of $\frac{m}{s}$ as expected for a velocity.

For the interval $4 < t < 5$, the velocity is unchanging. This means that $\frac{\Delta v}{t} = 0$, so we simply plot 0 for the whole interval on the acceleration graph.

Now we look at $5 < t < 8$ and see that velocity decreases by 9 in a time of 3 seconds. Thus the slope of the graph is $a = \frac{\Delta v}{t} = \frac{-9}{3} = -3$ so we plot that for the whole segment on the acceleration graph.

The Position Graph

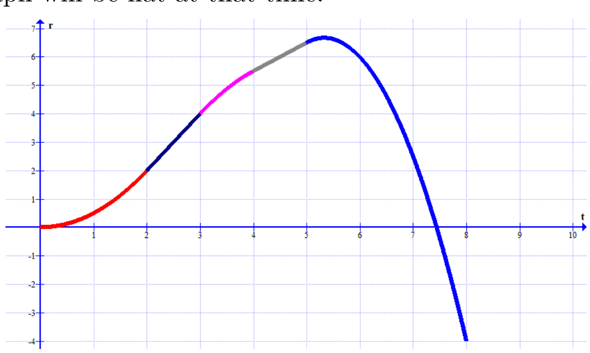
Everything is pretty easy when we consider only acceleration and velocity. It gets slightly harder when we consider position because quadratic functions show up in addition to lines. First, remember that $\Delta r = v_{avg}t$ (the definition of average velocity). Then remember that for a linear function, the average value of that function is just the midpoint of the segment. This is really all we need, but we can use the acceleration as a quick check. Acceleration controls which direction the parabola opens. Positive opens up, negative opens down, and 0 means that we have a linear position rather than quadratic for that interval.

Let's look again at the first interval $(0 < t < 2)$ the velocity on that interval begins at 0 and increases linearly to 2, so the average velocity is 1. We multiply that by the length of the interval in time $t = 2$ to find that we should have the point $(t = 2, r = 2)$ on our graph. Note that the velocity and position having the same numerical value is coincidental (we would not have had the same numerical value for most other choices of values) and meaningless (they don't even have the same units, so we cannot compare them in a useful way). Regardless of all that, we plot the point on the graph. Now we must consider how to connect them. We can either use that v is getting more positive, or equivalently that a is positive to get that we need an upward opening parabola. We graph that (the red segment in the position graph).

Now we look at the next segment where the acceleration doesn't change, $2 < t < 3$. On this segment, $a = 0$, so we should have a straight line for the position graph. This is confirmed by the fact that the velocity is constant on the velocity graph. We may then plot a straight line with slope 2 for the one unit interval to arrive at the point $(t = 3, a = 4)$. Another way is to find the area under the velocity graph during that time to see that Δr should increase by $2 \cdot 1 = 2$ units, then connect the dots with a straight line because velocity is constant. Both approaches are easy here. On a harder problem one or the other might present some advantage in ease of application, but both will necessarily give the same answer. This is the blue segment of the position graph.

I will do one more region, for more practice. Then I will finish the graph without more commentary. The next interval we can take is $3 < t < 4$. On this interval, acceleration is negative, so we get a downward opening parabola, but velocity is positive, so we are still increasing. To get the exact increase amount, we take the area under that segment. By inspecting the graph, the area is 1.5. Thus we have $\Delta r = 1.5$. This give the next point on the graph as $(t = 4, r = 5.5)$. We plot that point and remember that we need a downward sloping parabola. In order to get the slope of the parabola to cleanly meet the next segment, we have to notice that the slope at the end of the interval should be 1.

Now I finish the graph. The only thing left to note is that the velocity graph crosses 0 between time 5 and 6. We need the exact time to find the area of the triangle under the axis to get the change in position. We use $\Delta v = at$ with $\Delta v = -1$ and $a = -3$ to get $t = \frac{1}{3}$ so we cross the axis at $t = 5 + \frac{1}{3}$. My position graph will be flat at that time.



I will quickly note here that we could have done this problem by matching the functions derivatives at each boundary. I won't go through that method here, but if you thought to do that, it will definitely work.

Motion Graph Scenarios

All that seemed like a lot of (easy to mess up) arithmetic and probably seems pretty abstract. Lets do this with a real scenario. Note that in a real scenario, are looking for the graphs to be shaped right, and for the graphs to match. We don't know the exact values, so the vertical axes are arbitrary. The horizontal axes are somewhat arbitrary, but we should label them with $t_1, t_2, t_3, ...$ just so we can line up the events on different graphs. In this example, I have taken specific time values for my axes[†]. Also, note that there will always be ambiguity here. You won't lose points as long as your interpretation is reasonable.

Here are some rules and pointers, in no particular order:

- All accelerations are constant unless something about the scenario would make highly unrealistic[‡]
- The times on the graphs must align.
- Clearly larger speeds, or accelerations should be further from the axis.
- The area under the curve should be reasonable.

Example Motion Graph Scenario

You are initially at rest eating a carrot. After some time, you realize that you haven't irritated your teachers in 3 whole minutes, so you begin to walk down the hallway. After walking for some time you spot your government teacher. You slow to a stop so you can shout something about your homework assignment to read the constitution being unconstitutional. You stay stopped for a time discussing constitutional law. You become bored so you continue walking down the hallway, but you haven't even reached a running pace yet when you get distracted by a shiny object and slow to a stop again. After using your forensic skills to determine that it is a candy wrapper, you begin sprinting in the same direction to go get some candy. Implications aside, we need to translate this into physics. I think it is almost always easiest to start these with the velocity graph. Here is the translated version along with the time the interval starts at on my graph

- You initially have velocity 0 for some amount of time ($t = 0$)
- Your velocity increases until you eventually reach a walking pace ($t = 1$)
- You have a constant velocity for a while ($t = 2$)
- Your velocity decreases to 0 ($t = 4$)
- Your velocity stays 0 for a while ($t = 5$)
- Your velocity increases to a higher speed this time ($t = 7$)
- Your velocity decreases rather than becoming constant ($t = 11$)
- You stop again for some time. ($t = 12$)
- Your velocity increases until it is at the highest values anywhere else on the graph ($t = 17$)

Remember those times are arbitrary. Probably they should be labeled by $t_1 ... t_n$

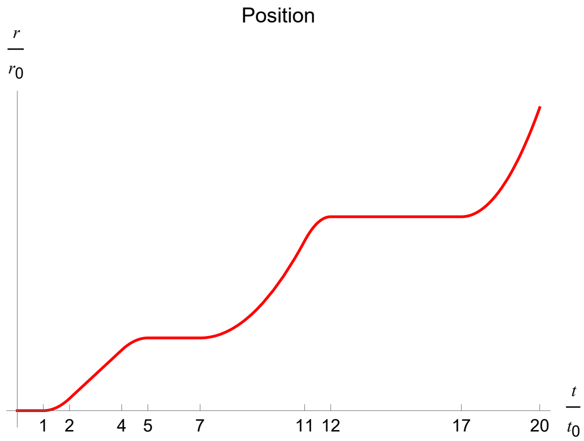
Now we can make that into a graph. See our graph on the next page.

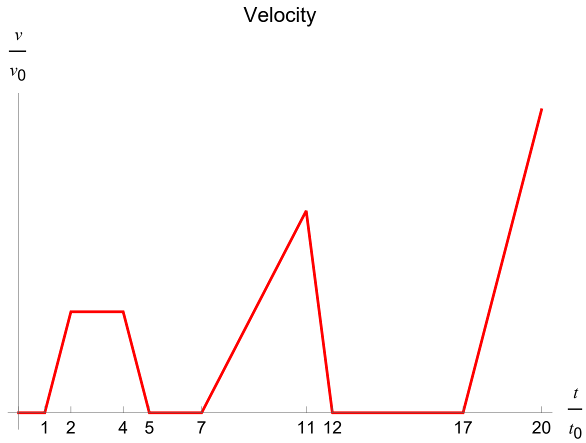
[†]because this lets me plot it on a computer... Computers don't do arbitrary. If it helps, feel free to replace the values on the x axis with t_1, t_2, t_3...

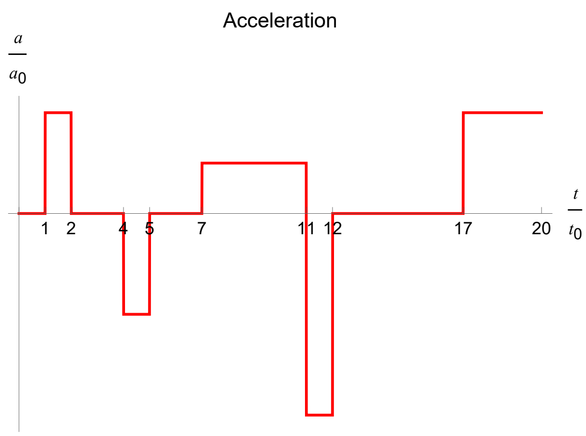
[‡]Usually this would be drag force, but also an asteroid falling toward the sun would have a changing gravity force, so constant acceleration would not make sense.

Once we have the velocity, we use the tools from section 5.1.3 and 5.1.3. To construct the position and acceleration. To summarize those rules with less math

- Positive slope on velocity graph means concave up position graph and positive acceleration
- Negative slope on velocity graph means concave down position graph and negative acceleration
- Flat velocity graph means straight position graph and 0 acceleration
- 0 velocity means constant position and 0 acceleration







If you want more practice translating between graphs, or translating english into graphs, a motion graph generator is available here. You are welcome to use it for extra practice.

5.1.4 Drag Forces

Now we have everything we need to analyze drag forces. We will need to use the rules from the previous sections, as well as (forces section citation here) so please review them if you don't feel familiar with them.

To build some intuition for drag force, we will use the familiar effects that come from air and water. Both of these are examples of drag forces. We will not consider syrupy liquids like honey here. Objects in these sort of liquids experience something called viscous drag that shares some properties with the drag force we will discuss here.

Grab a piece of paper. You can probably guess that it will accelerate downward much less rapidly than g and probably have some idea that this is due to air resistance, but what controls how strong air resistance is? Let's try some experiments.

If we crumple up the paper and drop it, it will fall much faster. We can intuit that drag force depends on area. You might guess surface area, but the surface area wasn't changed, whatever drawings fit before are still there now. So it must be cross sectional area: the area of a slice of the object perpendicular to its direction of motion. We don't really care about what area is involved here, so don't spend too much time pondering this.

Now, consider sprinting through air and through deep water. If you have ever tried sprinting through water it is really hard. We know water is much more dense than air, so we might guess that the density of the fluid is involved. If you think that it might be something to do with the polarity of water, imagine running through canola oil. It won't be much different.

Now notice how slightly crumpled paper actually falls when dropped. Notice that it fairly quickly reaches a velocity that appears constant and falls with that velocity afterward. This must mean that it is in equilibrium. Let's draw a free body diagram.

Since the only two forces are drag and gravity, drag must initially be less than gravity, but quickly becomes (nearly) equal as the object speeds up. Thus we intuit that drag depends on speed.

We now have a set of variables that we know are important, through intuition or experiment but we don't know the equation that relates those variables. This is a classic problem for dimensional analysis. Indeed, we did this problem in section 2.3.3 and got

$$F_d \propto \rho v^2 A \tag{5.1}$$

Remember that the proportionality is a reminder that dimensional analysis is only good up to a constant. We want to know what that constant might depend on. To figure that out, think about a bunch of fast moving objects that humans have created. They tend to be relatively pointy. That isn't an accident, pointy things typically experience lower drag than less pointy things.[§]. So we guess that the dimensionless constant is based on the objects geometry, so we get the equation

$$F_d = \frac{C}{2}\rho v^2 A \tag{5.2}$$

The factor of $\frac{1}{2}$ here is rather arbitrary. We could have pulled that into our constant C, but no one does.

For problems where we don't change fluids and the object's geometry doesn't change, we have a much simpler form

$$F_d \propto v^2 \tag{5.3}$$

[§]This is a major oversimplification of course, but it is certainly true that mounting a sheet of plywood to the front of your car will increase drag

5.1.5 Drag Force Graphs

Analyzing this system with equation would require calculus (and not particularly easy calculus), so we are going to use our motion graph skills instead. The rules are similar, but you will need to be more clever with the application. Here are some tips and comments:

- We will never care about values for these graphs, only the shape.
- It is almost always best to think about very long and very short times first, since the behavior is usually clearest there.
- Don't start with position!
- When you get stuck on one graph, try switching to another.
- If you are stuck on all of them, try another range of times to see what you can get from that
- A FBD can be helpful if you ever feel lost.

Drag Force Graph Example

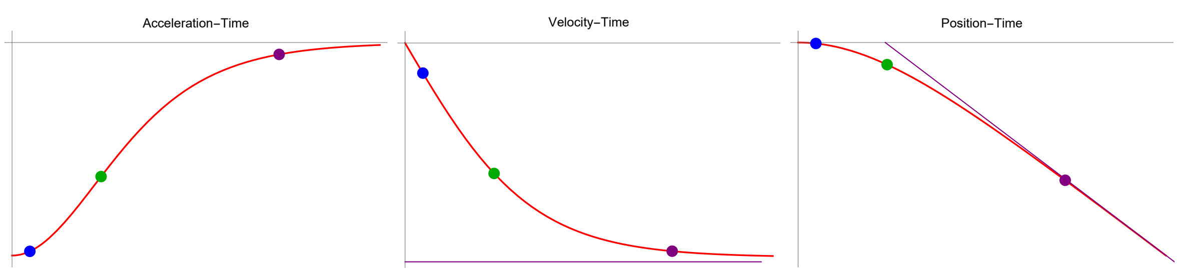
Consider the case where we have an object that is initially at rest at the top of a very tall building. At time 0, the object is dropped. Plot the position, velocity, and acceleration graphs of the object.

Start by choosing a sign convention. I am taking up positive. Note that the top of each graph is 0, so the graphs show only the negative region.

We will start by making two observations and considering the implications

- (before blue dot) At very short times, drag isn't important because $F_d \propto v^2$ and $v \approx 0$ so the force is small
  - The acceleration graph will be about g for short times
  - The velocity graph will be approximately linear with slope g for short times
  - The position graph will be concave down
- (after purple dot) At very long times, drag will balance gravity (you might wonder why drag never exceeds gravity. The answer is that if it did, we would slow down!) A bit more thought would show that the net force becomes less and less as we approach force balance, meaning we approach it asymptotically
  - The acceleration graph must asymptote to 0
  - The velocity graph must asymptote to some finite value (called the terminal speed, v_t)
  - The position graph will have a diagonal asymptote with slope v_t

Once we have plotted that, we need to think about the harder part. We look at the plot we have so far



Currently we have the behavior before the blue dots and the behavior after the purple dots. Now we want to get the behavior between the blue and green dots and between the green and purple dots. We make some more observations

- (blue to a bit before green) When the acceleration begins to deviate from constant, it must do so smoothly.
  - The acceleration graph is concave up
  - The velocity will keep decreasing, but will decrease less quickly
  - The position graph will continue to decrease more rapidly, but slightly less than quadratically now
- (a bit before green to purple) The acceleration graph is currently concave up and must transition to concave down. Also, velocity must never decrease faster than it did at an earlier time.
  - The acceleration graph must approach a straight line (at the green point)
  - Velocity cannot have a concavity shift.
  - Position graph will begin to approach the asymptote.

This concludes the drag force graphs.

5.2 Kinematics

Nothing here yet...

Chapter 6

Fluids

6.1 Pressure

Pressure is often defined as the force per unit area on a surface. This is certainly true and can be taken as one definition.

$$\boxed{P := \frac{F}{A}} \tag{6.1}$$

The problem you might have with this as the definition is that it isn't obvious how to apply it to pressure in a fluid. We can remedy this by saying that is the force per unit area that **would** exist on a surface if it was placed at that location in the fluid. * It will be useful to us to note that pressure has units of energy density. We will then make use of the ideal gas law

$$PV = nRT$$

We divide through by volume and note that the number of atoms N can be expressed as Avogadro's number (N_a) multiplied by the number of moles n to get

$$P = \frac{N}{N_a V} RT$$

$\frac{N}{V}$ is the number density of particles and is equal to $\frac{\rho}{\mu}$ (where ρ is density and μ is molecular mass) so this becomes

$$P = \frac{\rho}{\mu N_a} RT$$

the gas constant R is defined as $N_a k$ where k is Boltzmann's constant

$$P = \frac{\rho}{\mu} kT$$

So at least for an ideal gas, pressure is proportional to the internal energy density† in the gas. In non ideal gases, this equality does not hold, but pressure is still related to the internal energy of the gas. Fairly generally it is possible to write

$$\boxed{P = C\rho_e}$$

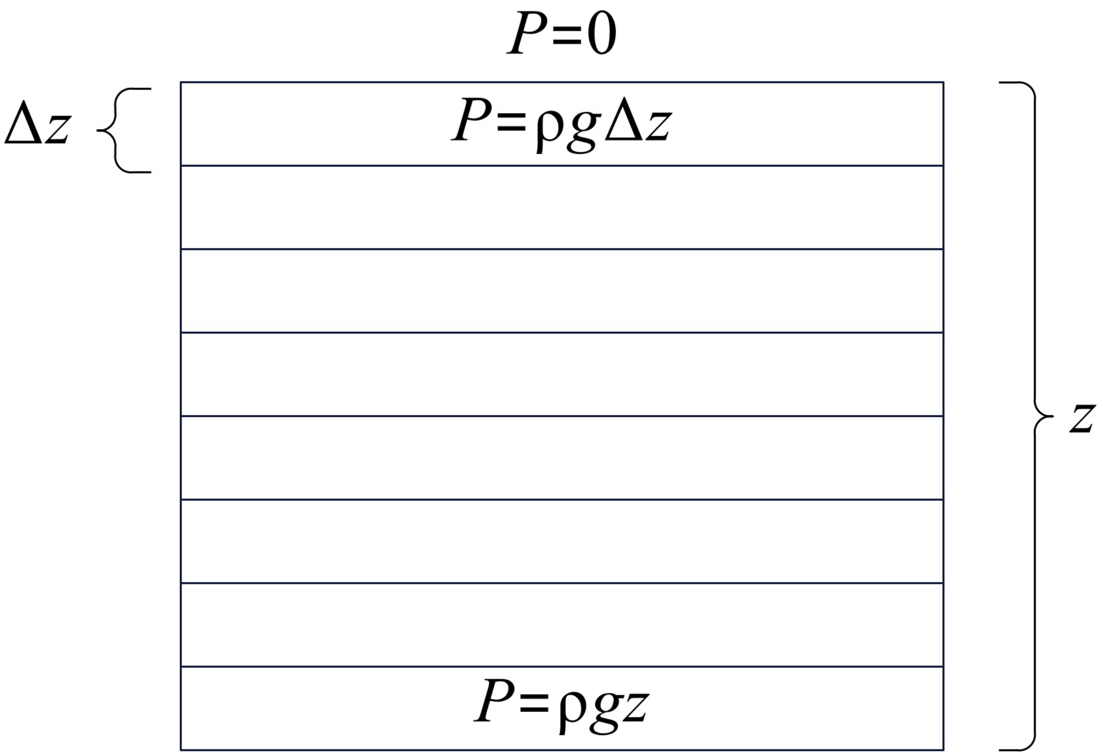
where C is some dimensionless constant and ρ_e is the internal energy density at that point in the fluid.

*If you find this definition unsatisfying, you could take the thermodynamic definition (which you will not need fo this class). This definition takes
$P := \frac{\partial E}{\partial V}$
The ∂ denotes a derivative with respect to a variable while all other variables are held constant.

†If you are wondering what internal energy is, it is energy from the state of the fluid, so it doesn't include stuff like gravitational potential or kinetic energies

6.1.1 Pressure in a fluid

Consider a large container filled with incompressible fluid with a density ρ. We'll ignore atmospheric pressure for the moment. ‡ Imagine that the water is composed of very thin "slices", each of width Δz. See figure



below

The top slice will have a pressure of 0 at the top. The bottom of the first slice (or the top of the second) must support the first slice, so we can use that to calculate the pressure there

$$M_s = \rho A \Delta z$$

so the downward force on the bottom of the first slice is

$$F = \rho A g \Delta z$$

we defined pressure as the force per unit area, so we divide by A to get

$$P_1 = \rho g \Delta z$$

at the bottom of the first slice. Note that the pressure will change across each slice.

The bottom of the second slice needs to support the first and second slices, so we could say that the pressure at the bottom of the second slide is

$$P_2 = 2\rho g \Delta z$$

Now lets say that the entire column had a height of z. Since we cut it into 8 pieces, $\Delta z = \frac{z}{8}$. The bottom of the last cell must support all 8 pieces, so the pressure must be

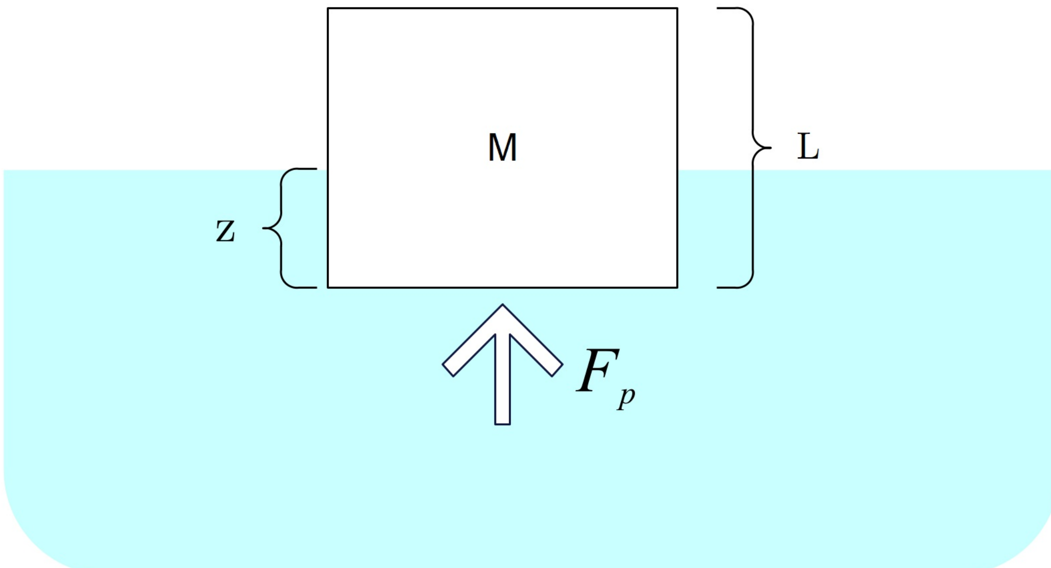
$$\boxed{P = \rho g z} \tag{6.2}$$

Since there was nothing special about our choice of z, this formula always gives the pressure at some distance below the surface of a static fluid.

‡If you want, you can think about ignoring atmospheric pressure as measuring the pressure relative to the atmosphere, this is called "gauge pressure" because most pressure gauges measure pressure relative to the atmosphere.

6.2 Buoyancy

We all have some idea that some objects float in water, while other sink. You may have noticed that more dense objects sink, while less dense objects float. Here we will explain the physics of this and try to expand. First, consider an object that has density ρ_b that is less than ρ_w as shown in the diagram.



We know it should float, but will it float mostly under the surface or mostly above the surface? To answer this, let's first assume that the body of water we are floating in is so large that we don't need to think about the change in water level[§] Consider the pressure force on the bottom of the block. The force should be given by

$$F_p = PA$$

Where A is the area of a cross section of the block. [¶] We know that the pressure in a fluid is given by $P = \rho g z$ where z is the distance below the surface. Then the pressure force on the bottom of our block (also called the **buoyant force**) should be

$$\boxed{F_p = \rho_w g z A} \tag{6.3}$$

We know that the block must be in force balance if it is not to accelerate. This condition gives that the pressure force balances gravity, or

$$Mg = \rho_w g z A$$

Which simplifies to

$$\boxed{M = \rho_w A z} \tag{6.4}$$

Since the right side of this equation is the mass of the displaced water, this tells us that **an object will float such that the mass of the displaced water is equal to the mass of the object.** This is called the Archimedes Principle. We can continue by using $M = \rho_b A L$ to get

$$\rho_b A L = \rho_w A z$$

We can cancel A and rearrange to finally arrive at

$$\rho_b L = \rho_w z$$

[§]This assumption has issues, but won't change the force based derivation used here.
[¶]Note that we don't need to think about the pressure on the sides since each side will cancel with the opposite side.

$$\boxed{\frac{z}{L} = \frac{\rho_b}{\rho_w}} \tag{6.5}$$

in words, the fraction of the object that is submerged is equal to the ratio of the density of the object to the density of water.

6.2.1 Sinking Objects

The discussion up to here was for floating objects. Sinking objects will have different behavior, specifically, the entire volume will be under the surface. **An object that is more dense than water will displace its entire volume of water.** This means the mass of the displaced water will be given by

$$M = \rho_w V$$

This means the buoyant force on a block that sinks will be

$$\boxed{F = \rho_w V g}$$

this must be less than the gravitational force on the object or the object would not sink.

6.3 Energy Conservation

6.3.1 Kinetic and Potential

Fluids still need to obey conservation laws. In particular, the energy of a fluid must be conserved as the fluid moves. For the purpose of this discussion, we will ignore any effects that dissipate energy.

First lets consider a small block that is dropped off a building. It can have two types of energy, gravitational potential (in the Earth block system) and kinetic. As the mass falls, one will convert to the other and our energy conservation equation will look like

$$mgh_1 + \frac{1}{2}mv_1^2 = mgh_2 + \frac{1}{2}mv_2^2$$

We have done this before, of course. Now we will imagine that the small block is composed of some liquid. Then we can write the mass of the object as

$$M = \rho V$$

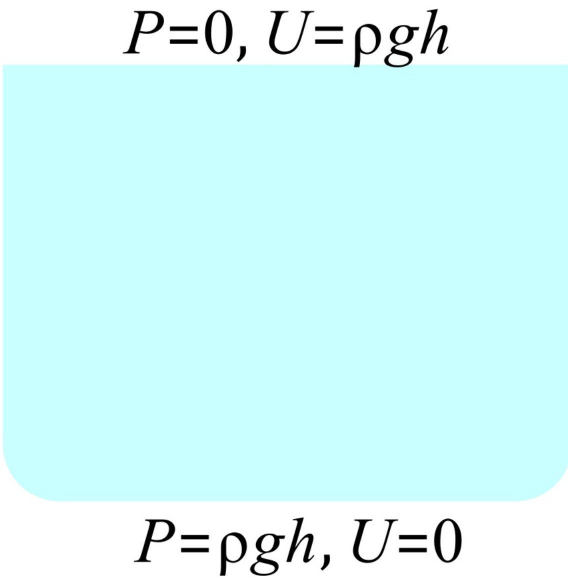
where V is the volume. Plugging that in and canceling V gives

$$\rho gh_1 + \frac{1}{2}\rho v_1^2 = \rho gh_2 + \frac{1}{2}\rho v_2^2$$

We could cancel ρhere, but that would lose us the clear interpretation of this as conservation of energy density.

6.3.2 Pressure in a static fluid

We already found that pressure is an energy density and that pressure is $P = \rho gh$, where h is the distance below the surface of the fluid. Now we want to figure out how pressure behaves in the energy density conservation equation.



If we take a region at the top of a pool, we can immediately say that its pressure is 0 (neglecting atmospheric pressure) and that it's gravitational potential energy density (denoted U here) is $U = \rho g h$. Using equation (6.2), from section 6.1.1 we know that the pressure at the bottom of the container is $P = \rho g h$ and it's internal energy was 0. Now imagine that we moved a small amount of water through the container very slowly (so that we can ignore KE). It's energy should not change as it moved. If it started at the top it would have had $U = \rho g h$ of potential energy and no pressure. At the end it would have had $P = \rho g h$ of pressure and not potential energy. It follows that

$$P_1 + \rho g h_1 = P_2 + \rho g h_2$$

Be very careful about how you apply this. Conservation of energy only guarantees that a fluids energy will be constant as it moves, not that fluid in two different regions must have the same energy. In general, two regions of a fluid may have different energies.

6.3.3 Bernoulli's equation

We have now looked at how energy conservation would work between kinetic and potential and between potential and pressure. It seems reasonable to combine them, and we can. This gives

$$\boxed{P_1 + \rho g h_1 + \frac{1}{2}\rho v^2 = P_2 + \rho g h_2 + \frac{1}{2}\rho v^2} \tag{6.6}$$

Once again, be careful to remember that this is a statement about a specific set of particles in the fluid as they move together, not about two different regions in a fluid.

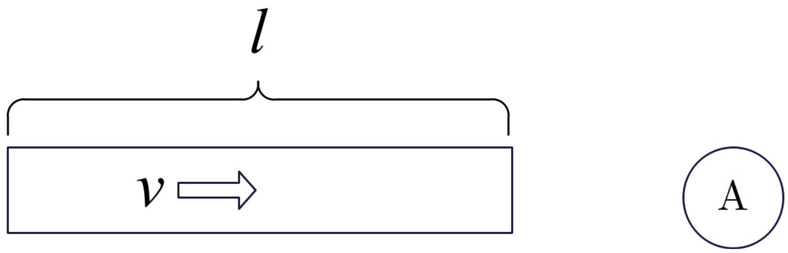
6.3.4 Mass Conservation

Obviously mass has been conserved in everything we have done in this class, but mass conservation was trivial before: we just had $M = M$ for any particle. Conservation of mass turns out to actually be useful for moving fluids. Basically, we can restate the law of conservation of mass as

"The same amount of mass flows through any part of a pipe in the same time."

We have made the assumption here that the pipe is not filling or emptying with time, and that the fluid is incompressible. For water pipes, these assumption are usually very accurate.

Now we just need to find an expression for the mass of the water that flows through a given cross section of pipe. In the diagram below, we see a pipe viewed from the side and viewed head on. The pipe section will be taken to have a length of 1m to make our argument easier to form.



The total mass of fluid in the pipe is given by $M = \rho V$. In our case $V = Al$ so we have

$$M = \rho A l$$

. The time required for this mass to flow through the end of the pipe is

$$t = \frac{l}{v}$$

This means that if we want the rate at which mass is flowing through we can divide the mass in the pipe by the time it takes to leave. This gives

$$\Phi = \frac{M}{t} = \frac{\rho A l}{\frac{l}{v}}$$

simplifying gives

$$\boxed{\Phi = \rho A v} \tag{6.7}$$

We can then imagine that the pipe changes size at some point. In this case all the mass that flows through the thick section must also flow through the thin section. This means that

$$\rho A_1 v_1 = \rho A_2 v_2$$

or, using that the density of an in compressible fluid may not change

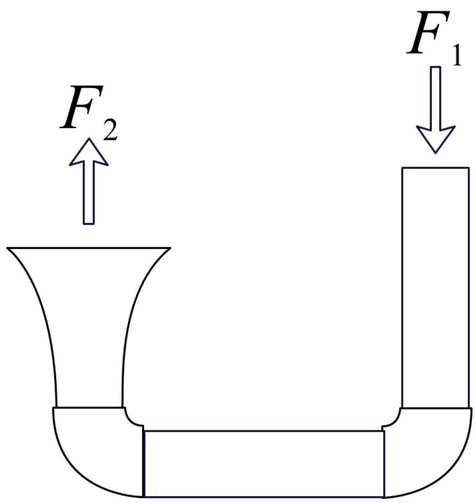
$$\boxed{A_1 v_1 = A_2 v_2} \tag{6.8}$$

Which, depending on the source, is known either as the **continuity equation**, or the **mass conservation equation**. Remember that we assumed that the pipe started full and remained full, if this is not the case then the mass that enters and the mass that leaves will not be the same.

6.4 Hydraulic Systems

The property that static fluids must always have the same pressure at the same height below the surface of the fluid has some counter-intuitive applications. Consider a system that consists of a tube filled with an incompressible fluid‖. A force $F_1$is applied downward on a small piston with area A_1. On the other side is a larger piston with area A_2. See diagram

‖Water is sometimes used here, but generally special hydraulic fluids work better



Lets take a system that is small enough that ρgh is very small compared to the pressure that comes from the pistons so we don't need to consider that.** Clearly the pressure must be

$$P = \frac{F_1}{A_1}$$

since the pressure is the same everywhere

$$F_2 = PA_2$$

but that means that (with a bit of algebra)

$$F_2 = \frac{A_2}{A_1} F_1 \tag{6.9}$$

in other words our device multiplied F_1 by the ratio of the area of the pistons. This ratio can be very large, so you might imagine that we could use this to do something like lift using a force that a human could generate. Indeed, lifting cars by hand is a standard application of hydraulic systems.

This all seems to have a problem though. It really seems like energy should not be conserved... Lets check that. The work done on the system by force F_1 as it moves a distance z_1 should be

$$W_1 = F_1 z_1$$

similarly the work done by the system on whatever we put on the larger piston should be

$$W_2 = F_2 z_2$$

we can use equation (6.9) to substitute in the value for F_2

$$W_2 = \frac{A_2}{A_1} F_1 z_2 \tag{6.10}$$

We know that the volume of an incompressible fluid cannot change. This means that the decrease in volume in the right section of pipe should equal the increase in volume of the left side. The decrease in volume of the right side is

$$\Delta V = A_1 z_1$$

The increase of the left should be

$$\Delta V = A_2 z_2$$

**This is extremely realistic because hydraulic systems are usually small and operate at very high pressure setting the equations equal and solving gives

$$z_2 = \frac{A_1}{A_2} z_1$$

Now we substitute this into equation (6.10) This gives

$$W_2 = \frac{A_2}{A_1} F_1 \frac{A_1}{A_2} z_1$$

Which of course simplifies to

$$W_2 = F_1 z_1$$

which was our original expression for W_1. Thus the system conserves energy. We can see from our analysis, that it would not conserve energy if it behaved in any other way...

Chapter 7

Oscillations and Waves

7.1 Oscillations

7.2 Basics of an oscillation

The most basic example of an oscillation is a mass on a spring set on a frictionless table. The mass will travel back and forth between two endpoints with some period. We know that when not extended or compressed, the spring will not provide any force. We call the position of the mass under this condition the equilibrium position. If we call the minimum position that the mass reaches during an oscillation $x = -A$, where A is the amplitude of the oscillation, then the equilibrium position will be $x = 0$ and the maximum position will be $x = A$.

7.2.1 Force and inertia

Let's say we start the mass at position $x = A$ (it's maximum position). Since the spring is extended, it will try to pull the mass back toward equilibrium. As a result, the mass will start to move with an increasing speed in the negative direction. The further it moves, the smaller the force on it will be. When it finally returns to equilibrium, the force disappears entirely.

As it moves through equilibrium, the force changes from pointing in the negative direction to pointing in the positive direction. Since the mass has momentum, the spring can't stop it instantly. As a result it overshoots. By symmetry, it won't stop until it reaches $x = -A$, at which time this whole process will repeat in the other direction.

7.2.2 Energy

Now we will look at the same thing using energy. Once again, we start with the spring totally extended. The mass will have no speed here, so all of it's energy is potential energy stored in the extension of the spring. The energy stored in the spring at max extension (or max compression) will be

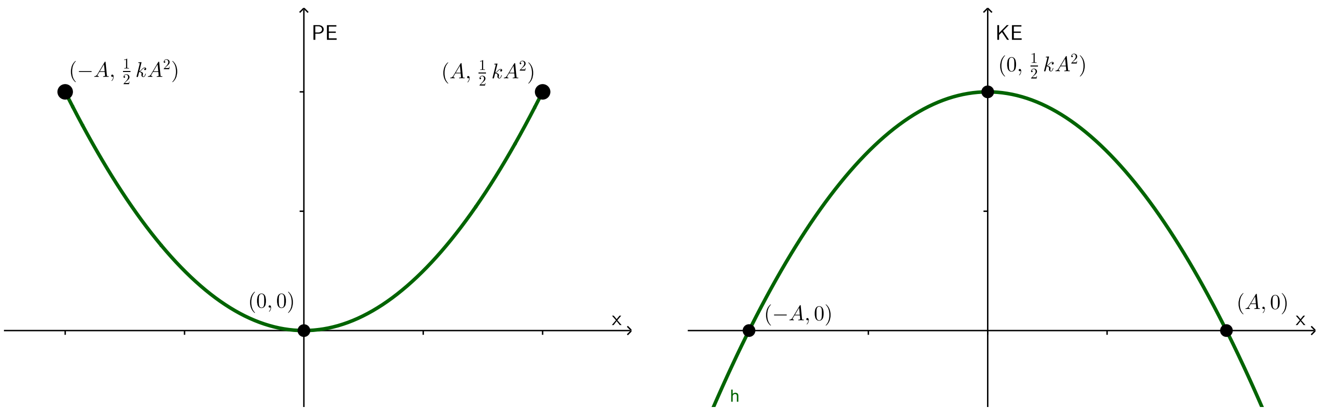
$$E_{tot} = \frac{1}{2}kA^2$$

At some arbitrary position, x, the energy in the spring will be

$$E_s = \frac{1}{2}kx^2$$

where x is the distance that the mass is from 0. We can plot the potential energy as a function of the distance of the object from equilibrium to get fig 7.1a

Figure 7.1: Energy Graphs



(a) The potential energy is a maximum when we have the largest possible extension, and will be given by $\frac{1}{2}kA^2$ there. When the mass is at the equilibrium point, there is no potential energy.

(b) The kinetic energy will have its largest value when the mass is at equilibrium. Here all the energy that was potential will be kinetic.

Now we can use that when we are at equilibrium the spring is uncompressed, so all potential energy has been changed to kinetic. The means the mass has all of the energy that started in the spring, so

$$E_{k,max} = E_{tot} = \frac{1}{2}kA^2$$

Since energy is conserved, the kinetic energy at some distance x from equilibrium can be expressed in terms of only the potential energy

$$E_k = E_{tot} - E_s$$

Rewriting this using the previous expressions

$$E_k = \frac{1}{2}kA^2 - \frac{1}{2}kx^2 = \frac{1}{2}k(A^2 - x^2)$$

Which we plot in figure 7.1b

Now, if we want to know the objects speed at some distance from equilibrium, we can find it using $E_k = \frac{1}{2}mv^2$ combined with the expression above.

$$\frac{1}{2}mv^2 = \frac{1}{2}k(A^2 - x^2)$$

$$v = \sqrt{\frac{k}{m}}\sqrt{A^2 - x^2}$$

7.2.3 Switching to time

Note that everything we have done so far was in terms of distance. This will help you understand how waves work when I explain them later. Now we would like to think in terms of time. The calculations require calculus, I do them in the appendix for those who are interested. For those who aren't, I will provide an intuitive way of thinking about it.

Here's what we can get from watching an oscillator or from what we have already seen.

1. We know that the position is periodic in time.

2. We know that the frequency is $f \propto \sqrt{\frac{k}{m}}$ (from dimensional analysis).

3. We know that the acceleration is proportional to displacement (so acceleration is periodic in time).

4. We know that velocity is periodic in time.

5. We know that the max displacement is A and the min displacement is -A.

Any time we see motion that is periodic like this, we should be thinking about sine and cosine curves.

Let's assume that we start measuring when the spring is at equilibrium. This means that $x = 0$ at $t = 0$ so let's use sine. The amplitude being A gives me the height of the sine wave, and the frequency being proportional to $\sqrt{\frac{k}{m}}$ tells me about the argument of the sine function. Our general sine function looks like

$$x = x_0 \sin(\omega t)$$

Using what we just got

$$x = A \sin\left(t\sqrt{\frac{k}{m}}\right)$$

Incidentally, this is where the $\frac{1}{2\pi}$ comes from in the frequency and period. The sine function doesn't cross 0 twice in a time of $\sqrt{m}k$, it instead takes $2\pi\sqrt{m}k$ to cross 0 twice, so we need to multiply that into the period.

We do the same thing with velocity that we did with position, except we start with v at it's max value. We know its max value from the kinetic energy argument given earlier.

$$v = \sqrt{\frac{k}{m}} A \cos\left(t\sqrt{\frac{k}{m}}\right)$$

Finally, $F = -kx$ or $ma = -kx$, so acceleration is exactly out of phase with displacement. Once again, we can solve it's max value using kinetic and potential energy

$$a = -\frac{k}{m} A \sin\left(t\sqrt{\frac{k}{m}}\right)$$

Now we want to go back to energy so we can express the kinetic and potential energy as a function of time.

$$KE = \frac{1}{2}mv^2 = \frac{1}{2}mA^2\frac{k}{m}\cos^2\left(t\sqrt{\frac{k}{m}}\right)$$

Potential energy is

$$PE = \frac{1}{2}kx^2 = \frac{1}{2}kA^2\sin^2\left(t\sqrt{\frac{k}{m}}\right)$$

And the total energy is

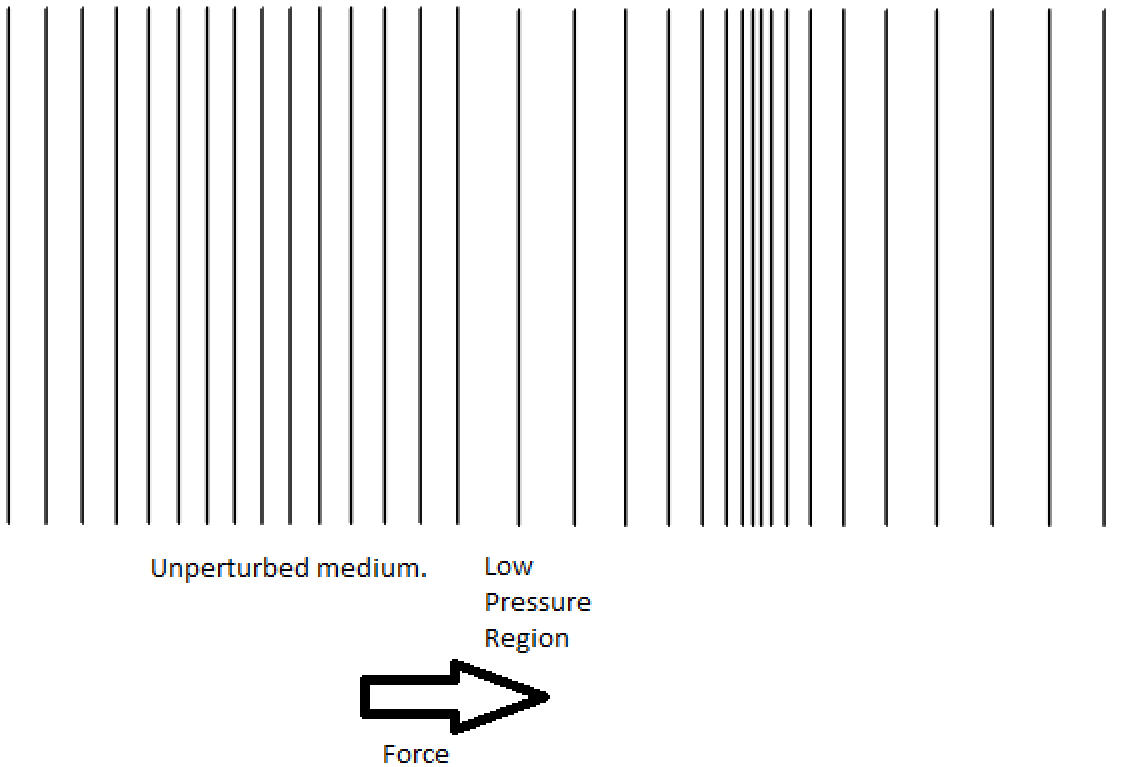
$$E_{tot} = PE + KE = \frac{1}{2}kA^2\left[\sin^2\left(t\sqrt{\frac{k}{m}}\right) + \cos^2\left(t\sqrt{\frac{k}{m}}\right)\right]$$

Since $\cos^2\theta + \sin^2\theta = 1$, this is constant in time and satisfies energy conservation.

7.3 Optional: Waves

Let's say we are standing on the x axis as a longitudinal wave goes by. We know that energy must be conserved, so let's look at the energy of a region that the wave passes through. Before the wave starts, we measure the particles close to us to have some pressure and be at some position. We'll call the region before the wave arrives the unperturbed medium and define those values as our equilibrium values.

Now a wave comes. Let's say that the wave arrives from the positive x direction, and that a pressure minimum arrives first. This will result in the particles starting to move from where they are now into the pressure hole created by the wave because the pressure difference creates a force. As they move I can measure that their kinetic energy increased. Since some of the particles just left my position to go into the nearby pressure minimum, I would also start to measure a lower density of particles, and thus the pressure where I am would start to decrease.



If I wait a while, eventually, the pressure in the hole will be the same as the pressure at my location. If the particles had no inertia, they would stop moving now, but they do have inertia, so they keep moving toward where the pressure hole used to be. As particles pile up at the position that used to be a pressure low, the pressure at that point becomes higher than the pressure where I am. This provides a force towards me, so the particles start to return.

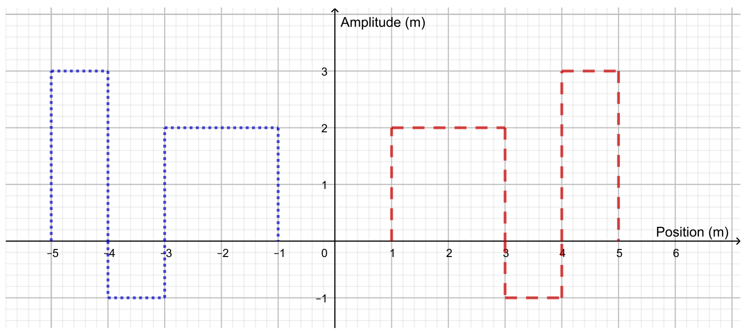
You may have noticed that pressure behaves a lot like the spring did in the previous case. It provides the restoring force that pushes particles back to equilibrium. Now let's look at how the energy varies through a cycle. As the wave starts to move through, the particles acquire kinetic energy as they move to fill the pressure hole. They have the highest speed (and thus highest KE) when the pressure is at equilibrium. Then they acquire more pressure energy as they compress and stop.

Thus the energy exhibits the same behavior that it did for the oscillator. It shouldn't come as a surprise that the equations also produce sine and cosine curves for displacement, velocity, and acceleration.

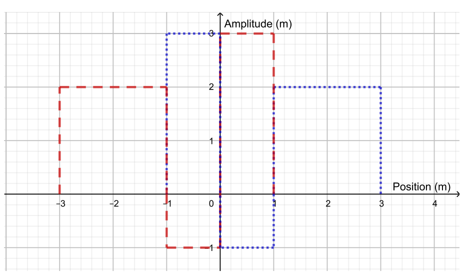
7.3.1 Wave Superposition

A given string, or a given particle in a medium, can have only one position at a time. This means that if multiple waves are propagating through the same medium, nature must have a way of determining where the particles should be. It does this in the simplest way possible: it just adds the current amplitude of each wave.

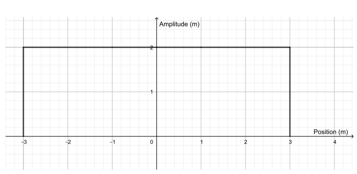
Take a string, lets say that we have two wave pulses that are propagating through it in opposite directions. The wave on the left is moving toward the right, and vis-versa If we could stop time and look at the waves, we would see something like this:



If we wait a time Δt, the waves will each have moved a distance $c\Delta t$ where c is the wave propagation speed. Let's say that the wave speed is 2 $\frac{\text{m}}{\text{s}}$. After 2 seconds, each wave will have gone 4 m. So the waves will look like this:



Now we simply add up the two waves at each point to get the result, which in this case is very nice:



Sine waves are added using the same method: just add up the amplitude of each wave at each point. Assuming that we have identical sine curves, waves that are shifted over exactly half a wavelength will add up to give 0 amplitude. With two identical sources, points where the waves have gone the same distance will be maxima, and points where one wave has traveled half a wavelength further than the other will be minima.

7.3.2 Standing Waves

When two sine waves with the same amplitude and frequency are traveling in opposite directions, they add up to give a standing wave. In a standing wave, there are points called nodes where no motion occurs, and points called anti-nodes, where the largest motion occurs.

7.3.3 Optional: Resonance

Tube and String Resonance

Tube and string resonance is a direct application of standing waves. When a wave reaches the end of a tube or string, it will reflect back. The way that it reflects depends on the boundary condition. The possible conditions are as follows

- Closed or Fixed End: No displacement, this means that the wave must reflect inverted so that the wave and its reflection will add up to 0 at the end.
- Open or Free End: Maximum displacement, * this means that the wave must reflect without a phase change, so that the wave and its reflection will add up to the largest value possible.

Imagine that we have a wave coming into a tube. When it reaches the end, it will reflect. Then when that reflection reaches the end, the reflection will reflect, and so on. In order to have a standing wave, we need the wave and all of its reflections to add up nicely. This will only be possible with very specific tube lengths.

Another way to look at the problem is to just apply the above boundary conditions to the standing wave itself. Either way, we get the following relations.

- 2 open ends (tube): $L = \frac{n\lambda}{2}$
- 2 fixed ends (string): $L = \frac{n\lambda}{2}$
- 1 fixed and 1 free end (tube or string): $L = \left(\frac{1}{4} + \frac{n}{2}\right)\lambda$

For 2 open ends or 1 fixed and 1 free end, n is the number of nodes. For 2 fixed ends, n is the number of anti-nodes.

7.3.4 Other systems

Many systems can exhibit resonance. For example, we discussed a mass on a spring earlier. If any force is added that is always in phase with the oscillation, the force will always be increasing the energy of the system. This means that even with very small forces, huge amplitudes can result. The only thing that prevents the energy of the system from becoming infinite is that any real physical system has mechanisms that dissipate energy. Some examples include friction and air resistance. We use the term 'damping' to refer to anything that takes energy out of a wave or oscillation.

*For the case of a sound wave, this can be understood by recognizing that the pressure of the room cannot be changed. This means that we must have a pressure node and thus a displacement anti-node.

Chapter 8

Appendices and Problem Solutions

8.1 Solutions: Energy

8.2 Solutions: Momentum

8.3 Solutions: Angular Momentum

8.4 Solutions: Gravity

8.5 Solutions: Forces

8.6 Solutions: Motion Graphs

8.7 Solutions: Kinematics

8.8 Solutions: Oscillators

8.8.1 Appendix, A calculus solution

We know that the force is given by

$$F = -kx$$

By Newton's 2nd law

$$ma = -kx$$

But acceleration is the second derivative of position, so we can write

$$\frac{d^2x}{dt^2} = -\frac{k}{m}x$$

Now we need a function that gives a negative copy of itself back as a second derivative. We can try either sine or cosine. Let's try

$$x = A\sin(\omega t)$$

The second derivative of this is

$$-A\omega^2 \sin(\omega t)$$

which we can write as

$$-\omega^2 x$$

If we throw this into our equation in place of the second derivative, we get

$$-\omega^2 x = -\frac{k}{m}x$$

This means that

$$\omega = \sqrt{\frac{k}{m}}$$

Which makes our solutions

$$x = A\sin\left(t\sqrt{\frac{k}{m}}\right)$$

$$v = A\sqrt{\frac{k}{m}}\cos\left(t\sqrt{\frac{k}{m}}\right)$$

$$a = -A\frac{k}{m}\sin\left(t\sqrt{\frac{k}{m}}\right)$$