Fluid Notes Draft

1 Fluids

1.1 Pressure

Pressure is often defined as the force per unit area on a surface. This is certainly true and can be taken as one definition.

$$\boxed{P := \frac{F}{A}} \tag{1}$$

The problem you might have with this as the definition is that it isn't obvious how to apply it to pressure in a fluid. We can remedy this by saying that is the force per unit area that **would** exist on a surface if it was placed at that location in the fluid. [1] It will be useful to us to note that pressure has units of energy density. We will then make use of the ideal gas law

$$PV = nRT$$

We divide through by volume and note that the number of atoms N can be expressed as Avogadro's number (N_a) multiplied by the number of moles n to get

$$P = \frac{N}{N_a V} RT$$

$\frac{N}{V}$ is the number density of particles and is equal to $\frac{\rho}{\mu}$ (where ρ is density and μ is molecular mass) so this becomes

$$P = \frac{\rho}{\mu N_a} RT$$

the gas constant R is defined as $N_a k$ where k is Boltzmann's constant

$$P = \frac{\rho}{\mu} kT$$

So at least for an ideal gas, pressure is proportional to the internal energy density[2] in the gas. In non ideal gases, this equality does not hold, but pressure is still related to the internal energy of the gas. Fairly generally it is possible to write

$$\boxed{P = C\rho_e}$$

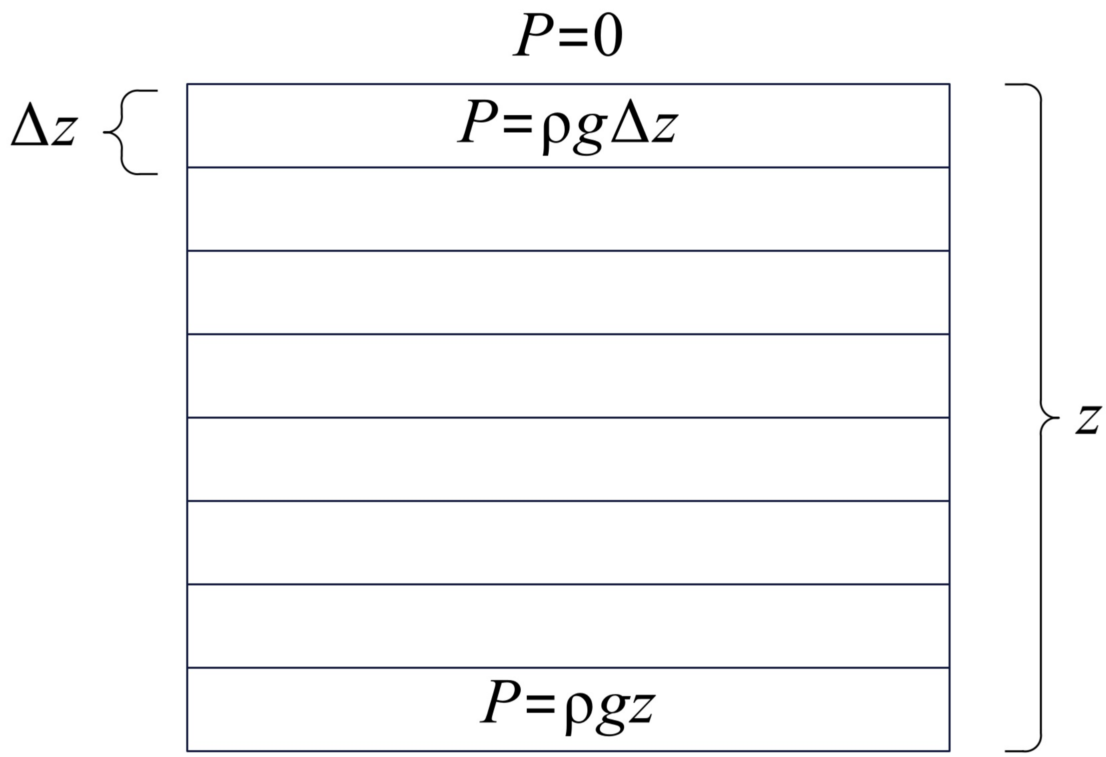
where C is some dimensionless constant and ρ_e is the internal energy density at that point in the fluid.

[1] If you find this definition unsatisfying, you could take the thermodynamic definition (which you will not need fo this class). This definition takes
$P := \frac{\partial E}{\partial V}$
The ∂ denotes a derivative with respect to a variable while all other variables are held constant.

[2] If you are wondering what internal energy is, it is energy from the state of the fluid, so it doesn't include stuff like gravitational potential or kinetic energies

1.2 Pressure in a fluid

Consider a large container filled with incompressible fluid with a density ρ. We'll ignore atmospheric pressure for the moment. [3] Imagine that the water is composed of very thin "slices", each of width Δz. See figure below



The top slice will have a pressure of 0 at the top. The bottom of the first slice (or the top of the second) must support the first slice, so we can use that to calculate the pressure there

$$M_s = \rho A \Delta z$$

so the downward force on the bottom of the first slice is

$$F = \rho A g \Delta z$$

we defined pressure as the force per unit area, so we divide by A to get

$$P_1 = \rho g \Delta z$$

at the bottom of the first slice. Note that the pressure will change across each slice.

The bottom of the second slice needs to support the first and second slices, so we could say that the pressure at the bottom of the second slide is

$$P_2 = 2\rho g \Delta z$$

Now lets say that the entire column had a height of z. Since we cut it into 8 pieces, $\Delta z = \frac{z}{8}$. The bottom of the last cell must support all 8 pieces, so the pressure must be

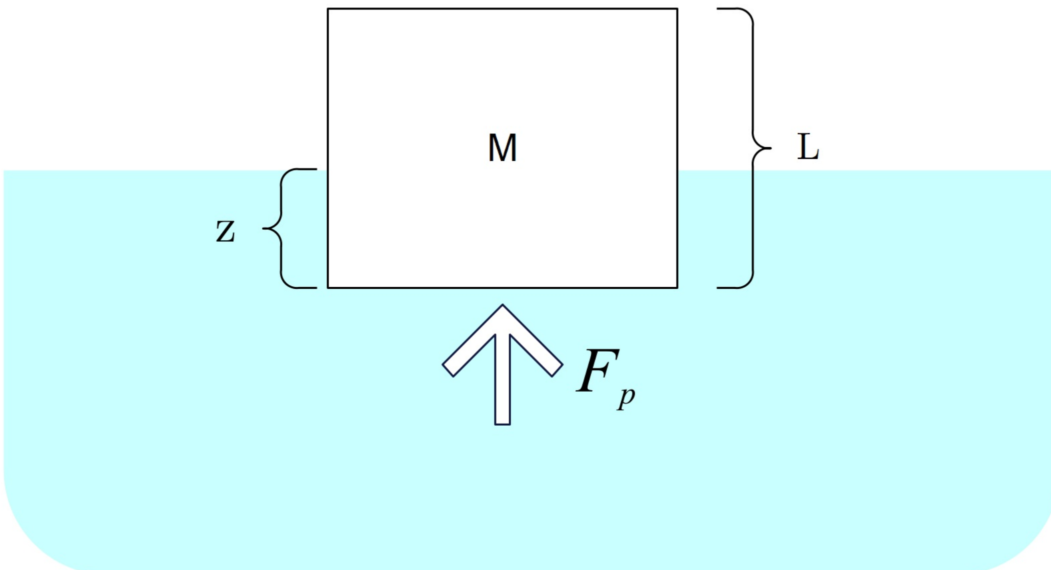
$$\boxed{P = \rho g z} \tag{2}$$

Since there was nothing special about our choice of z, this formula always gives the pressure at some distance below the surface of a static fluid.

[3] If you want, you can think about ignoring atmospheric pressure as measuring the pressure relative to the atmosphere, this is called "gauge pressure" because most pressure gauges measure pressure relative to the atmosphere.

1.3 Buoyancy

We all have some idea that some objects float in water, while other sink. You may have noticed that more dense objects sink, while less dense objects float. Here we will explain the physics of this and try to expand. First, consider an object that has density ρ_b that is less than ρ_w as shown in the diagram.



We know it should float, but will it float mostly under the surface or mostly above the surface? To answer this, let's first assume that the body of water we are floating in is so large that we don't need to think about the change in water level[4] Consider the pressure force on the bottom of the block. The force should be given by

$$F_p = PA$$

Where A is the area of a cross section of the block. [5] We know that the pressure in a fluid is given by $P = \rho g z$ where z is the distance below the surface. Then the pressure force on the bottom of our block (also called the **buoyant force**) should be

$$\boxed{F_p = \rho_w g z A} \tag{3}$$

We know that the block must be in force balance if it is not to accelerate. This condition gives that the pressure force balances gravity, or

$$Mg = \rho_w g z A$$

Which simplifies to

$$\boxed{M = \rho_w A z} \tag{4}$$

Since the right side of this equation is the mass of the displaced water, this tells us that **an object will float such that the mass of the displaced water is equal to the mass of the object.** This is called the Archimedes Principle. We can continue by using $M = \rho_b A L$ to get

$$\rho_b A L = \rho_w A z$$

We can cancel A and rearrange to finally arrive at

$$\rho_b L = \rho_w z$$

[4]This assumption has issues, but won't change the force based derivation used here.

[5]Note that we don't need to think about the pressure on the sides since each side will cancel with the opposite side.

$$\boxed{\frac{z}{L} = \frac{\rho_b}{\rho_w}} \tag{5}$$

in words, the fraction of the object that is submerged is equal to the ratio of the density of the object to the density of water.

1.3.1 Sinking Objects

The discussion up to here was for floating objects. Sinking objects will have different behavior, specifically, the entire volume will be under the surface. **An object that is more dense than water will displace its entire volume of water.** This means the mass of the displaced water will be given by

$$M = \rho_w V$$

This means the buoyant force on a block that sinks will be

$$\boxed{F = \rho_w V g}$$

this must be less than the gravitational force on the object or the object would not sink.

1.4 Energy Conservation

1.4.1 Kinetic and Potential

Fluids still need to obey conservation laws. In particular, the energy of a fluid must be conserved as the fluid moves. For the purpose of this discussion, we will ignore any effects that dissipate energy.

First lets consider a small block that is dropped off a building. It can have two types of energy, gravitational potential (in the Earth block system) and kinetic. As the mass falls, one will convert to the other and our energy conservation equation will look like

$$mgh_1 + \frac{1}{2}mv_1^2 = mgh_2 + \frac{1}{2}mv_2^2$$

We have done this before, of course. Now we will imagine that the small block is composed of some liquid. Then we can write the mass of the object as

$$M = \rho V$$

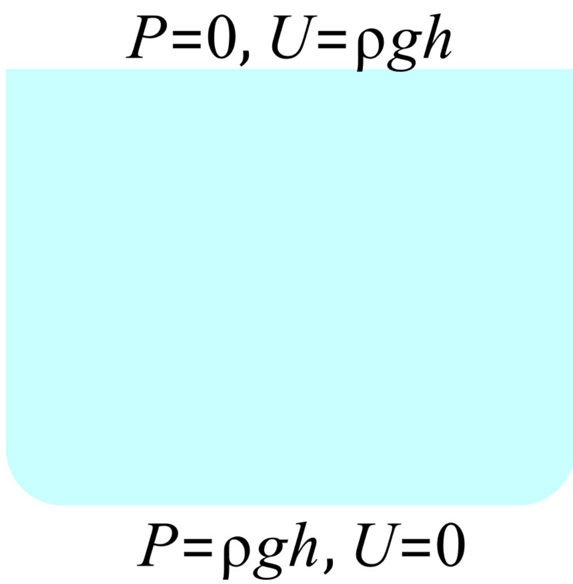
where V is the volume. Plugging that in and canceling V gives

$$\rho gh_1 + \frac{1}{2}\rho v_1^2 = \rho gh_2 + \frac{1}{2}\rho v_2^2$$

We could cancel ρhere, but that would lose us the clear interpretation of this as conservation of energy density.

1.4.2 Pressure in a static fluid

We already found that pressure is an energy density and that pressure is $P = \rho gh$, where h is the distance below the surface of the fluid. Now we want to figure out how pressure behaves in the energy density conservation equation.



If we take a region at the top of a pool, we can immediately say that its pressure is 0 (neglecting atmospheric pressure) and that it's gravitational potential energy density (denoted U here) is $U = \rho g h$. Using equation (2), from section 1.2 we know that the pressure at the bottom of the container is $P = \rho g h$ and it's internal energy was 0. Now imagine that we moved a small amount of water through the container very slowly (so that we can ignore KE). It's energy should not change as it moved. If it started at the top it would have had $U = \rho g h$ of potential energy and no pressure. At the end it would have had $P = \rho g h$ of pressure and not potential energy. It follows that

$$P_1 + \rho g h_1 = P_2 + \rho g h_2$$

Be very careful about how you apply this. Conservation of energy only guarantees that a fluids energy will be constant as it moves, not that fluid in two different regions must have the same energy. In general, two regions of a fluid may have different energies.

1.4.3 Bernoulli's equation

We have now looked at how energy conservation would work between kinetic and potential and between potential and pressure. It seems reasonable to combine them, and we can. This gives

$$\boxed{P_1 + \rho g h_1 + \frac{1}{2}\rho v^2 = P_2 + \rho g h_2 + \frac{1}{2}\rho v^2} \tag{6}$$

Once again, be careful to remember that this is a statement about a specific set of particles in the fluid as they move together, not about two different regions in a fluid.

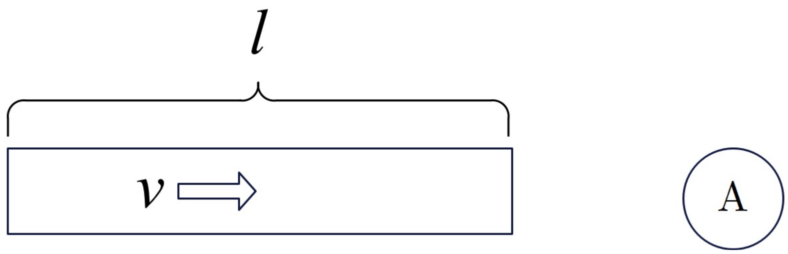
1.5 Mass Conservation

Obviously mass has been conserved in everything we have done in this class, but mass conservation was trivial before: we just had $M = M$ for any particle. Conservation of mass turns out to actually be useful for moving fluids. Basically, we can restate the law of conservation of mass as

"The same amount of mass flows through any part of a pipe in the same time."

We have made the assumption here that the pipe is not filling or emptying with time, and that the fluid is incompressible. For water pipes, these assumption are usually very accurate.

Now we just need to find an expression for the mass of the water that flows through a given cross section of pipe. In the diagram below, we see a pipe viewed from the side and viewed head on. The pipe section will be taken to have a length of 1m to make our argument easier to form.



The total mass of fluid in the pipe is given by $M = \rho V$. In our case $V = Al$ so we have

$$M = \rho Al$$

. The time required for this mass to flow through the end of the pipe is

$$t = \frac{l}{v}$$

This means that if we want the rate at which mass is flowing through we can divide the mass in the pipe by the time it takes to leave. This gives

$$\Phi = \frac{M}{t} = \frac{\rho Al}{\frac{l}{v}}$$

simplifying gives

$$\boxed{\Phi = \rho Av} \tag{7}$$

We can then imagine that the pipe changes size at some point. In this case all the mass that flows through the thick section must also flow through the thin section. This means that

$$\rho A_1 v_1 = \rho A_2 v_2$$

or, using that the density of an in compressible fluid may not change

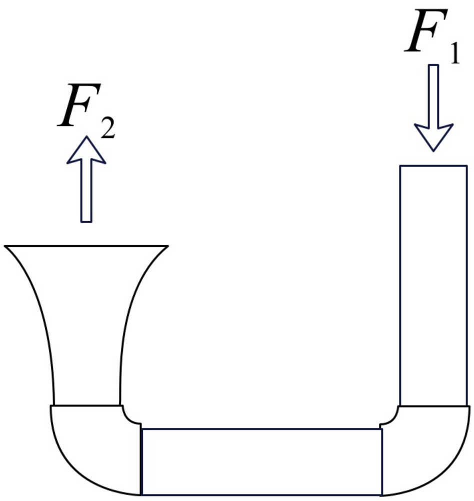
$$\boxed{A_1 v_1 = A_2 v_2} \tag{8}$$

Which, depending on the source, is known either as the **continuity equation**, or the **mass conservation equation**. Remember that we assumed that the pipe started full and remained full, if this is not the case then the mass that enters and the mass that leaves will not be the same.

1.6 Hydraulic Systems

The property that static fluids must always have the same pressure at the same height below the surface of the fluid has some counter-intuitive applications. Consider a system that consists of a tube filled with an incompressible fluid[6]. A force $F_1$is applied downward on a small piston with area A_1. On the other side is a larger piston with area A_2. See diagram

[6] Water is sometimes used here, but generally special hydraulic fluids work better



Lets take a system that is small enough that ρgh is very small compared to the pressure that comes from the pistons so we don't need to consider that.[7] Clearly the pressure must be

$$P = \frac{F_1}{A_1}$$

since the pressure is the same everywhere

$$F_2 = PA_2$$

but that means that (with a bit of algebra)

$$F_2 = \frac{A_2}{A_1}F_1 \tag{9}$$

in other words our device multiplied F_1 by the ratio of the area of the pistons. This ratio can be very large, so you might imagine that we could use this to do something like lift using a force that a human could generate. Indeed, lifting cars by hand is a standard application of hydraulic systems.

This all seems to have a problem though. It really seems like energy should not be conserved... Lets check that. The work done on the system by force F_1 as it moves a distance z_1 should be

$$W_1 = F_1 z_1$$

similarly the work done by the system on whatever we put on the larger piston should be

$$W_2 = F_2 z_2$$

we can use equation (9) to substitute in the value for F_2

$$W_2 = \frac{A_2}{A_1}F_1 z_2 \tag{10}$$

We know that the volume of an incompressible fluid cannot change. This means that the decrease in volume in the right section of pipe should equal the increase in volume of the left side. The decrease in volume of the right side is

$$\Delta V = A_1 z_1$$

The increase of the left should be

$$\Delta V = A_2 z_2$$

[7]This is extremely realistic because hydraulic systems are usually small and operate at very high pressure

setting the equations equal and solving gives

$$z_2 = \frac{A_1}{A_2} z_1$$

Now we substitute this into equation (10) This gives

$$W_2 = \frac{A_2}{A_1} F_1 \frac{A_1}{A_2} z_1$$

Which of course simplifies to

$$W_2 = F_1 z_1$$

which was our original expression for W_1. Thus the system conserves energy. We can see from our analysis, that it would not conserve energy if it behaved in any other way...